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Classification of classical twists of the standard Lie bialgebra structure on a loop algebra

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ABSTRACT

The standard Lie bialgebra structure on an affine Kac–Moody algebra induces a Lie bialgebra structure on the underlying loop algebra and its parabolic subalgebras. In this paper we classify all classical twists of the induced Lie bialgebra structures in terms of Belavin–Drinfeld quadruples up to a natural notion of equivalence. To obtain this classification we first show that the induced bialgebra structures are defined by certain solutions of the classical Yang–Baxter equation (CYBE) with two parameters. Then, using the algebro–geometric theory of CYBE, based on torsion free coherent sheaves, we reduce the problem to the well-known classification of trigonometric solutions given by Belavin and Drinfeld. The classification of twists in the case of parabolic subalgebras allows us to answer recently posed open questions regarding the so-called quasi-trigonometric solutions of CYBE.

1. Introduction

A Lie bialgebra is a pair (L, δ) consisting of a Lie algebra L and a linear map $\delta\colon L \longrightarrow L \otimes L$, called Lie cobracket, inducing a compatible Lie algebra structure on the dual space $L^\vee$. This notion originated in [11] as the infinitesimal counterpart of a Poisson Lie group. Shortly after, in [12,13], Lie bialgebras were described as quasi-classical limits of certain quantum groups and received a fundamental role in the quantum group theory.

Having a Lie bialgebra structure δ on a Lie algebra L we can obtain new Lie bialgebra structures using a procedure called twisting. More precisely, let t be a skew-symmetric tensor in $L \otimes L$ satisfying

$$\mathrm{CYB}(t) = \mathrm{Alt}((\delta \otimes 1)t),$$

where

$$\mathrm{CYB}(t) := [t^{12}, t^{13}] + [t^{12}, t^{23}] + [t^{13}, t^{23}],$$
$$\mathrm{Alt}(x_1 \otimes x_2 \otimes x_3) := x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_3 \otimes x_1 + x_3 \otimes x_1 \otimes x_2$$

and, for example, $[(a \otimes b)^{12}, (c \otimes d)^{23}] := a \otimes [b, c] \otimes d$. Then the linear map $\delta_t := \delta + dt$ is a Lie bialgebra structure on L. Such a tensor t is called a classical twist.

The most important example of a Lie bialgebra structure is the standard structure δ on a symmetrizable Kac–Moody algebra $\mathfrak{K} := \mathfrak{K}(A)$ introduced in [13]. In the case when the Cartan matrix A is of finite type or, equivalently, when $\mathfrak{K}$

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is a finite-dimensional semi-simple Lie algebra, the standard structure δ and all its twisted versions δ_t are known to be quasi-triangular, i.e. they are of the form dr for some $r \in \mathfrak{K} \otimes \mathfrak{K}$ satisfying the classical Yang–Baxter equation $\mathrm{CYB}(r) = 0$.

When the matrix A is of affine type, the standard structure on $\mathfrak{K}$ induces a Lie bialgebra structure on $[\mathfrak{K}, \mathfrak{K}]/Z(\mathfrak{K})$, where $Z(\mathfrak{K})$ is the centre of $\mathfrak{K}$, which we will also call standard. The latter Lie algebra is known (see [23]) to be isomorphic to the loop algebra $\mathfrak{L}^\sigma$ over a simple finite-dimensional Lie algebra $\mathfrak{g}$ corresponding to an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ of finite order m. It has the following explicit description

$$\mathfrak{L}^\sigma = \left\{ f \in \mathfrak{g}[z, z^{-1}] \mid f(\varepsilon_\sigma z) = \sigma(f(z)) \right\}, \qquad \varepsilon_\sigma := \exp(2\pi i/m).$$

We denote the induced standard Lie bialgebra structure on $\mathfrak{L}^\sigma$ by δ_0^σ and call its twists $\delta_t^\sigma := \delta_0^\sigma + dt$ twisted standard structures. These Lie bialgebra structures are not quasi-triangular, but pseudoquasitriangular as is shown in Theorem 3.3, i.e. they are defined by meromorphic functions $r\colon \mathbb{C}^2 \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$, also known as r-matrices, satisfying the two-parametric classical Yang–Baxter equation (CYBE)

$$\mathrm{CYB}(r)(x_1, x_2, x_3) := [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = 0.$$

For example, the trigonometric r-matrix given by a Belavin–Drinfeld (BD) quadruple Q, corresponding to an outer automorphism $\nu \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ (see [4]), gives rise to a twisted standard structure δ_Q^σ on $\mathfrak{L}^\sigma$ for any finite order automorphism σ whose coset is conjugate to $\nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. It turns out that any r-matrix defining a twisted standard bialgebra structure on $\mathfrak{L}^\sigma$ is globally holomorphically equivalent to a trigonometric solution in the sense of the Belavin–Drinfeld classification (see Theorem 3.4). We refer to such r-matrices as σ-trigonometric.

We call two twisted standard structures δ_t^σ and δ_s^σ (regularly) equivalent if there is a function

$$\phi \in \mathrm{Aut}_{\mathbb{C}[z^m, z^{-m}]-\mathrm{LieAlg}}(\mathfrak{L}^\sigma) = \left\{ f\colon \mathbb{C}^* \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g}) \mid f \text{ is regular and } f(\varepsilon_\sigma z) = \sigma f(z)\sigma^{-1} \right\},$$

called a regular equivalence, such that $\delta_t^\sigma \phi = (\phi \otimes \phi)\delta_s^\sigma$. The main result of this paper is the classification of twisted standard structures up to regular equivalence. The classification is obtained by reducing our problem to the classification of trigonometric r-matrices up to holomorphic equivalence given in [4]. To deal with the difference between the notions of equivalence we use the geometric formalism of CYBE presented in [7]. More precisely, one of the key results in [7] is that certain coherent sheaves of Lie algebras on Weierstraß cubic curves give rise to so-called geometric r-matrices, satisfying a geometric version of CYBE. In Section 5 we prove the following extension property:

Theorem A. *A formal equivalence of geometric r-matrices at the smooth point at infinity of the Weierstraß cubic curve gives rise to an isomorphism of the corresponding sheaves of Lie algebras.*

It is shown in [1] that all σ-trigonometric r-matrices arise as geometric r-matrices from coherent sheaves of Lie algebras on the nodal Weierstraß cubic with section $\mathfrak{L}^\sigma$ on the set of smooth points. Since holomorphic equivalences are formal, this result and Theorem A show that the notions of holomorphic and regular equivalence for σ-trigonometric r-matrices coincide (see Theorem 5.8). In particular, this leads to the desired classification.

Theorem B. *For any twisted standard structure δ_t^σ there is a regular equivalence ϕ of $\mathfrak{L}^\sigma$ and a BD quadruple $Q = \left(\Gamma_1, \Gamma_2, \gamma, t_{\mathfrak{h}}\right)$ such that*

$$\delta_t^\sigma \phi = (\phi \otimes \phi)\delta_Q^\sigma.$$

Furthermore, if $Q' = (\Gamma_1', \Gamma_2', \gamma', t_{\mathfrak{h}}')$ is another BD quadruple, then the twisted bialgebra structures δ_Q^σ and $\delta_{Q'}^\sigma$ are regularly equivalent if and only if there is an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(\Gamma_i) = \Gamma_i'$ for $i = 1, 2$, $\vartheta\gamma\vartheta^{-1} = \gamma'$ and $(\vartheta \otimes \vartheta)t_{\mathfrak{h}} = t_{\mathfrak{h}}'$.

Since the notions of holomorphic and regular equivalence for σ-trigonometric r-matrices coincide, the second part of Theorem B can be refined in the following way: two σ-trigonometric r-matrices r_Q^σ and $r_{Q'}^\sigma$, given by BD quadruples Q and Q', are holomorphically equivalent if and only if there is a Dynkin diagram automotphism γ, such that $\gamma(Q) = Q'$ (see Theorem 5.10 for details). This fact was stated without a proof in [4, §6.4, Remark 4], and we are not aware of any other proof

Let Π^σ be the set of simple roots of $\mathfrak{L}^\sigma$, $S \subsetneq \Pi^\sigma$ and $\mathfrak{p}_+^S \subseteq \mathfrak{L}^\sigma$ be the corresponding parabolic subalgebra (see Section 2.2). The standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ restricts to a Lie bialgebra structure on the parabolic subalgebra $\mathfrak{p}_+^S$. We refer to this Lie bialgebra structure as the restricted standard structure.

In the special case $\sigma = \mathrm{id}$ and $S = \Pi^{\mathrm{id}} \setminus \{\widetilde{\alpha}_0\}$, where $\widetilde{\alpha}_0$ is the affine root of $\mathfrak{L}^{\mathrm{id}} = \mathfrak{g}[z, z^{-1}]$, the classical twists of the restricted standard structure are in one-to-one correspondence with so-called quasi-trigonometric solutions of CYBE. Such r-matrices were studied and classified in terms of BD quadruples in [26,32]. We use this classification in Section 4.3 to demonstrate the first part of Theorem B in the special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. This connection to quasi-trigonometric solutions serves as a motivation for our study of restricted standard structures.

We discover that Theorem B also gives a full classification of classical twists of restricted Lie bialgebra structures. More formally, for any classical twist $t \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$ the structure of $\mathfrak{L}^\sigma$ guarantees that the regular equivalence between δ_t^σ and some δ_Q^σ, given by Theorem B, can be chosen to fix the parabolic subalgebra $\mathfrak{p}_+^S$. The following theorem summarizes this observation.

Theorem C. *For any classical twist $t \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$ of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ there exists a regular equivalence ϕ that restricts to an automorphism of $\mathfrak{p}_+^S$ and a BD quadruple $Q = \left(\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h}\right)$ such that*

$$\Gamma_1 \subseteq S \ \text{ and } \ \delta_t^\sigma \phi = (\phi \otimes \phi)\delta_Q^\sigma.$$

Let $Q' = (\Gamma_1', \Gamma_2', \gamma', t_\mathfrak{h}')$, $\Gamma_1' \subseteq S$, be another BD quadruple. A regular equivalence between twisted standard structures δ_Q^σ and $\delta_{Q'}^\sigma$ restricts to an automorphism of $\mathfrak{p}_+^S$ if and only if the induced Dynkin diagram automorphism ϑ preserves S, i.e. $\vartheta(S) = S$.

The theorems stated above provide us with a list of interesting consequences:

- Letting $\sigma = \mathrm{id}$ and $S = \Pi^{\mathrm{id}} \setminus \{\widetilde{\alpha}_0\}$ in Theorem C we obtain an alternative proof of the classification of all quasi-trigonometric solutions [26,32];
- The necessary and sufficient condition to have an id-trigonometric r-matrix which is not regularly equivalent to a quasi-trigonometric one is the existence of a BD quadruple $(\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ such that for any automorphism ϑ of the extended Dynkin diagram of $\mathfrak{g}$ we have $\widetilde{\alpha}_0 \in \vartheta(\Gamma_1)$. Analysing Dynkin diagrams, we conclude that any id-trigonometric r-matrix is regularly equivalent to a quasi-trigonometric one if and only if $\mathfrak{g}$ is of type A_n, C_n, B_{2-4} or D_{4-10};
- We have mentioned that any σ-trigonometric r-matrix is holomorphically equivalent to a trigonometric one in the sense of the Belavin–Drinfeld classification. Combining the structure theory of $\mathfrak{L}^\sigma$ (Section 2.2) and Theorem B we can improve that result and get more control over that equivalence. More precisely, let ν be an outer automorphism of $\mathfrak{g}$, σ be a finite order automorphism of $\mathfrak{g}$ whose coset is conjugate to $\nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and r_t^σ be the σ-trigonometric r-matrix defining a twisted standard Lie bialgebra structure δ_t^σ on $\mathfrak{L}^\sigma$. Applying to r_t^σ the regular equivalence, given by Theorem B, and regrading to the principle grading, i.e. grading corresponding to the Coxeter automorphism $\sigma_{(\mathbf{1};|\nu|)}$, we obtain a trigonometric r-matrix X depending on the quotient of its parameters:

$$r_t^\sigma(x, y) \xmapsto{\text{regular eq.}} r_Q^\sigma(x, y) \xmapsto{\text{regrading}} r_Q^{\sigma(\mathbf{1};|\nu|)}(x, y) = X(x/y);$$

- We answer questions one and two posed at the end of [8] concerning an explicit formula for the quasi-trigonometric solution given by a BD quadruple Q and its connection with the trigonometric solution described by the same quadruple Q (see [4]);
- We prove the quasi-trigonometric version of Drinfeld's conjecture on rational r-matrices (see [33,34]): any quasi-trigonmetric solution is polynomially equivalent to a quasi-trigonometric solution of the form

$$\frac{yC}{x-y} + p(x) + q(y),$$

where $C \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir element, $p, q \in (\mathfrak{g} \otimes \mathfrak{g})[z]$ and $\deg(p), \deg(q) \leq 1$.

Similar to the σ-trigonometric case, rational r-matrices (in the sense of [33,34]) arise as geometric r-matrices (see [7]). Applying Theorem A in the framework of rational solutions we see that the notion of polynomial equivalence in the works [33,34] by Stolin coincides with the notion of holomorphic equivalence considered in [4] by Belavin and Drinfeld. We collect these auxiliary results in Appendix.

In [29] Montaner, Stolin and Zelmanov classified all Lie bialgebra structures on $\mathfrak{g}[z]$ by classifying classical twists within all possible Drinfeld double algebras. One of the main points in their argument is the aforementioned classification of quasi-trigonometric solutions [32] or, equivalently, the classification of classical twists within one of the doubles. From this perspective, our work is a natural step towards the classification of all Lie bialgebra structures on $\mathfrak{L}^{\mathrm{id}} = \mathfrak{g}[z, z^{-1}]$ or, more generally, $\mathfrak{L}^\sigma$.

2. Preliminaries

In this section we give a brief review of the theory of Lie bialgebras and loop algebras as well as set up notation and terminology used throughout the paper. Most of the presented results on Lie bialgebras can be found in [10,14] and [27]. A detailed exposition of the theory of loop algebras can be found in [9,23] and [20, §X.5].

2.1. Lie bialgebras, manin triples and twisting

A *Lie coalgebra* is a pair (L, δ) consisting of a vector space L over a field k of characteristic zero and a linear map $\delta: L \longrightarrow L \otimes L$, called *Lie cobracket*, such that for all $x \in L$

$$\delta(x) + \tau\delta(x) = 0 \ \text{ and } \ \mathrm{Alt}((\delta \otimes 1)\delta(x)) = 0, \tag{2.1}$$

where $\tau(x_1 \otimes x_2) := x_2 \otimes x_1$ and $\mathrm{Alt}(x_1 \otimes x_2 \otimes x_3) := x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_3 \otimes x_1 + x_3 \otimes x_1 \otimes x_2$. These conditions guarantee that the restriction of the dual map $\delta^\vee: (L \otimes L)^\vee \longrightarrow L^\vee$ to $L^\vee \otimes L^\vee$ defines a Lie algebra structure. A *morphism* between two Lie coalgebras (L, δ) and $\left(L', \delta'\right)$ is a linear map $\phi: L \longrightarrow L'$ such that

$$(\phi \otimes \phi)\delta = \delta'\phi. \tag{2.2}$$

A *Lie bialgebra* is a triple[1] $(L, [-,-], \delta)$ such that $(L, [-,-])$ is a Lie algebra, (L, δ) is a Lie coalgebra and the following compatibility condition holds

$$\delta([x, y]) = x \cdot \delta(y) - y \cdot \delta(x) \qquad \forall x, y \in L, \tag{2.3}$$

where $x \cdot (y_1 \otimes y_2) := [x, y_1] \otimes y_2 + y_1 \otimes [x, y_2]$. In other words, δ is a 1-cocycle of L with values in $L \otimes L$. A linear map between two Lie bialgebras is a *Lie bialgebra morphism* if it is a morphism of both Lie algebra and Lie coalgebra structures.

Lie bialgebras are closely related to *Manin triples*, i.e. triples (L, L_+, L_-), where L is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form B and $L_\pm$ are isotropic subalgebras of L with respect to that form, such that $L = L_+ \dotplus L_-$.[2] The definition immediately implies that $L_\pm$ are Lagrangian subalgebras of L which are paired non-degenerately by B. We say that two Manin triples (L, L_+, L_-) and $\left(L', L'_+, L'_-\right)$ are *isomorphic* if there is a Lie algebra isomorphism $\phi\colon L \longrightarrow L'$ such that

$$\phi(L_\pm) = L'_\pm \ \text{ and } \ B(x, y) = B(\phi(x), \phi(y)) \ \text{ for all } x, y \in L. \tag{2.4}$$

Every Lie bialgebra (L, δ) gives rise to the Manin triple $\left(L \dotplus L^\vee, L, L^\vee\right)$ with the canonical bilinear form B given by

$$B(x + f, y + g) := f(y) + g(x) \qquad \forall x, y \in L, \ \forall f, g \in L^\vee, \tag{2.5}$$

and the Lie algebra structure on $L \dotplus L^\vee$ defined by

$$[x, f] := \mathrm{ad}^*_x f + (f \otimes 1)(\delta(x)) \qquad \forall x \in L, \ \forall f \in L^\vee, \tag{2.6}$$

where $\mathrm{ad}^*_x := -\mathrm{ad}^\vee_x$ is the coadjoint action.

Remark 2.1. The Lie algebra structure (2.6) is the unique Lie algebra structure on $L \dotplus L^\vee$ making the canonical form B invariant and L, $L^\vee$ into Lagrangian subalgebras. The space $L \dotplus L^\vee$ equipped with this particular Lie algebra structure is called the *classical double* of (L, δ). ⋄

The converse statement is not true, i.e. not every Manin triple (L, L_+, L_-) induces a Lie bialgebra structure on L_+. However, this is the case when the dual map $[-,-]^\vee\colon L_-^\vee \longrightarrow (L_- \otimes L_-)^\vee$ of the Lie bracket on L_- restricts to a map $\delta\colon L_+ \longrightarrow L_+ \otimes L_+$, where we use the injection $L_+ \longrightarrow L_-^\vee$ induced by B. This condition can be equivalently formulated in the following way: there is a linear map $\delta\colon L_+ \to L_+ \otimes L_+$ such that

$$B(\delta(x), y \otimes z) = B(x, [y, z]) \qquad \forall x \in L_+, \ \forall y, z \in L_-. \tag{2.7}$$

When this condition is satisfied, we say that the Manin triple (L, L_+, L_-) *defines* the Lie bialgebra (L_+, δ).

Remark 2.2. Let ϕ be an isomorphism between two Manin triples $M = (L, L_+, L_-)$ and $M' = \left(L', L'_+, L'_-\right)$. If M defines a Lie bialgebra structure (L_+, δ), then M' also defines a Lie bialgebra structure (L'_+, δ') and $\phi|_{L_+}\colon (L_+, \delta) \longrightarrow (L'_+, \delta')$ is a Lie bialgebra isomorphism. ⋄

Remark 2.3. Generally, there may exist many non-isomorphic Manin triples defining the same Lie bialgebra structure. However, in the finite-dimensional case the condition (2.7) holds automatically and the correspondence between Manin triples and Lie bialgebras described above is one-to-one. ⋄

Having a Lie bialgebra structure, we can produce a new bialgebra structure by means of a procedure called *twisting*. Let (L, δ) be a Lie bialgebra and $t \in L \otimes L$ be a skew-symmetric tensor satisfying the identity

$$\mathrm{CYB}(t) = \mathrm{Alt}\left((\delta \otimes 1)t\right), \tag{2.8}$$

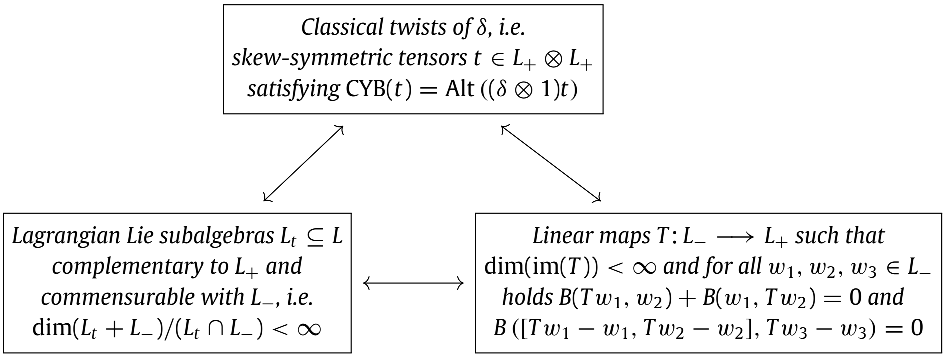
where $\mathrm{CYB}(t) := [t^{12}, t^{13}] + [t^{12}, t^{23}] + [t^{13}, t^{23}]$. Then the linear map $\delta_t := \delta + dt$, where $dt(x) := x \cdot t$ for all $x \in L$, defines a new Lie bialgebra structure on L. The skew-symmetric tensor t is called a *classical twist* of δ.

It was implicitly shown in [26,32–34] that the problem of classification of classical twists of some particular Lie bialgebra structures can be reduced to the classification of Lagrangian Lie subalgebras. In the following theorem we summarize and generalize these ideas.

Theorem 2.4. *Let (L_+, δ) be a Lie bialgebra defined by the Manin triple (L, L_+, L_-). Then there are the following one-to-one correspondences:*

[1] For convenience, the notation $[-,-]$ for the Lie bracket on L will be omitted from the triple.

[2] We write $A \dotplus B$ (or $A \oplus B$) meaning the direct sum of A and B as vector spaces (modules), but not as Lie algebras. The latter is denoted by $A \times B$.



Proof. Let $t = x_i \otimes y^i \in L_+ \otimes L_+$ be a classical twist[3] of δ. Define the linear map $T\colon L_- \longrightarrow L_+$ and the subspace $L_t \subseteq L$ by

$$T := B(y^i, -)x_i \;\text{ and }\; L_t := \{Tw - w \mid w \in L_-\}\,. \tag{2.9}$$

We now show that they meet the requirements of the theorem. The conditions $\dim(\mathrm{im}(T)) < \infty$ and $L_+ \dotplus L_t = L$ hold by definition. For all $w_1, w_2 \in L_-$ we have

$$\begin{aligned} B\,(Tw_1 - w_1, Tw_2 - w_2) &= -B(Tw_1, w_2) - B(w_1, Tw_2) \\ &= -B(y^i, w_1)B(x_i, w_2) - B(y^i, w_2)B(x_i, w_1). \end{aligned} \tag{2.10}$$

Therefore, the skew-symmetry of t is equivalent to the skew-symmetry of T and to L_t being a Lagrangian subspace. To prove the commensurability of L_t and L_- we note that $\ker(T) = L_t \cap L_-$ and hence

$$\dim\,(L_-/(L_t \cap L_-)) = \dim\,(\mathrm{im}(T))\,. \tag{2.11}$$

This shows that $L_t \cap L_-$ has finite codimension inside L_-. The commensurability now follows from the fact that L_- has codimension at most $\dim(\mathrm{im}(T))$ inside $L_t + L_-$. Finally, the last condition follows from the identity

$$B\,(w_1 \otimes w_2 \otimes w_3, \mathrm{CYB}(t) - \mathrm{Alt}\,((\delta \otimes 1)\,t)) = -B\,([Tw_1 - w_1, Tw_2 - w_2], Tw_3 - w_3)\,, \tag{2.12}$$

where $w_1, w_2, w_3 \in L_-$. This identity is obtained by repeating the argument in the proof of [25, Theorem 7] within our framework.

Conversely, given a Lagrangian Lie subalgebra $L' \subseteq L$, satisfying the conditions of the theorem, we define the linear map $T\colon L_- \longrightarrow L_+$ in the following way: any $w \in L_-$ can be uniquely written as $w_+ + w'$, for some $w_+ \in L_+$ and $w' \in L'$; We let $T(w) := w_+$. Then $L' = \{Tw - w \mid w \in L_-\}$ and the commensurability of L' and L_- implies that the rank of T is finite. The other two conditions on T hold because of the relations (2.10) and the Lagrangian property of L'. To construct the classical twist $t \in L_+ \otimes L_+$ we note that B gives a non-degenerate pairing between the finite-dimensional spaces $L_-/\ker(T)$ and $\mathrm{im}(T)$. Let $\{Tw_i\}_{i=1}^n$ be a basis for $\mathrm{im}(T)$ and $\left\{v^i + \ker(T)\right\}_{i=1}^n$ be its dual basis for $L_-/\ker(T)$. Then

$$B(w_k, -Tv^i)Tw_i = B(Tw_k, v^i)Tw_i = Tw_k, \tag{2.13}$$

for all $k \in \{1, \dots, n\}$. Since T is completely determined by its action on $\{w_i\}_{i=1}^n$, we have the equality $T = -B(Tv^i, -)Tw_i$. We define $t := -Tw_i \otimes Tv^i$. The identities (2.12) and (2.10) guarantee that t meets the desired requirements and $L' = L_t$. ■

Remark 2.5. It follows that if (L_+, δ) is a Lie bialgebra defined by the Manin triple (L, L_+, L_-) and t is a classical twist of δ, then the twisted Lie bialgebra $(L_+, \delta + dt)$ is defined by the Manin triple (L, L_+, L_t). Equivalently,

$$B\,(\delta(x) + x \cdot t, (Tw_1 - w_1) \otimes (Tw_2 - w_2)) = B\,(x, [Tw_1 - w_1, Tw_2 - w_2])\,, \tag{2.14}$$

for all $x \in L_+$ and $w_1, w_2 \in L_-$. ◇

2.2. Loop algebras

Let $\mathfrak{g}$ be a fixed finite-dimensional simple Lie algebra over $\mathbb{C}$ and σ be an automorphism of $\mathfrak{g}$ of finite order $|\sigma| \in \mathbb{Z}_+$. The eigenvalues of σ are $\varepsilon_\sigma^k := e^{2\pi i k/|\sigma|}$, $k \in \mathbb{Z}$, and we have the following $\mathbb{Z}/|\sigma|\mathbb{Z}$-gradation of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{k=0}^{|\sigma|-1} \mathfrak{g}_k^\sigma\,, \tag{2.15}$$

[3] We use the Einstein summation convention: $x_i \otimes y^i = \sum_i x_i \otimes y^i$.

where $\mathfrak{g}_k^\sigma$ is the eigenspace of σ corresponding to the eigenvalue ε_σ^k. The tensor product

$$\mathfrak{L} := \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] = \bigoplus_{k\in\mathbb{Z}} z^k\mathfrak{g} = \left\{f : \mathbb{C}^* \longrightarrow \mathfrak{g} \mid f \text{ is regular}\right\}, \tag{2.16}$$

equipped with the bracket described by $[z^i x, z^j y] := z^{i+j}[x, y]$, for all $x, y \in \mathfrak{g}$ and $i, j \in \mathbb{Z}$, is a $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$. *The loop algebra* $\mathfrak{L}^\sigma$ over $\mathfrak{g}$ is the $\mathbb{Z}$-graded Lie subalgebra of $\mathfrak{L}$ defined by

$$\mathfrak{L}^\sigma := \bigoplus_{k\in\mathbb{Z}} z^k\mathfrak{g}_k^\sigma = \{f \in \mathfrak{L} \mid \sigma(f(z)) = f(\varepsilon_\sigma z)\}, \tag{2.17}$$

where $\mathfrak{g}_{k+\ell|\sigma|}^\sigma = \mathfrak{g}_k^\sigma$ for all $\ell \in \mathbb{Z}$. It possesses an invariant non-degenerate symmetric bilinear form B, which is given by

$$B(f, g) := \operatorname{res}_{z=0}\left[\frac{1}{z}\kappa(f(z), g(z))\right] \qquad \forall f, g \in \mathfrak{L}, \tag{2.18}$$

where κ stands for the Killing form on $\mathfrak{g}$.

Remark 2.6. We can extend σ to an automorphism on $\mathfrak{L}$ by $\sigma(z^k x) := (z/\varepsilon_\sigma)^k\sigma(x)$. Then $\mathfrak{L}^\sigma$ can be viewed as the Lie subalgebra of $\mathfrak{L}$ consisting of fixed points of the extended action of σ on $\mathfrak{L}$. In particular, we have the identity $\mathfrak{L} = \mathfrak{L}^{\mathrm{id}}$. This motivates our choice of notation. $\diamond$

2.2.1. Structure theory (outer automorphism case)

The classification of all finite order automorphisms of $\mathfrak{g}$, explained in [20,22,23], gives the following relation at the level of loop algebras: for any finite order automorphism σ there is an automorphism ν of $\mathfrak{g}$, induced by an automorphism of the corresponding Dynkin diagram, such that $\mathfrak{L}^\sigma \cong \mathfrak{L}^\nu$. Therefore, we first describe the structure of $\mathfrak{L}^\nu$ and then explain how regrading of $\mathfrak{L}^\nu$ carries over the structure theory to $\mathfrak{L}^\sigma$.

Let $\mathfrak{g} = \mathfrak{n}'_- \dotplus \mathfrak{h}' \dotplus \mathfrak{n}'_+$ be a triangular decomposition of $\mathfrak{g}$ and $\widetilde{\nu}$ be an automorphism of the corresponding Dynkin diagram. The induced outer automorphism ν of $\mathfrak{g}$ is described explicitly by

$$\nu(x_i^\pm) = x_{\widetilde{\nu}(i)}^\pm, \qquad \nu(h_i) = h_{\widetilde{\nu}(i)}, \tag{2.19}$$

where $\{x_i^-, h_i, x_i^+\}$ is a fixed set of standard Chevalley generators for $\mathfrak{g}$. The order of such an automorphism is necessarily 1, 2 or 3. The subalgebra $\mathfrak{g}_0^\nu$ turns out to be simple with the following triangular decomposition

$$\mathfrak{g}_0^\nu = \underbrace{(\mathfrak{g}_0^\nu \cap \mathfrak{n}'_-)}_{=:\,\mathfrak{n}_-} \dotplus \underbrace{(\mathfrak{g}_0^\nu \cap \mathfrak{h}')}_{=:\,\mathfrak{h}} \dotplus \underbrace{(\mathfrak{g}_0^\nu \cap \mathfrak{n}'_+)}_{=:\,\mathfrak{n}_+}. \tag{2.20}$$

Moreover, when $|\nu| = 2$ or 3 the subspace $\mathfrak{g}_1^\nu$ is an irreducible $\mathfrak{g}_0^\nu$-module. In the case $|\nu| = 3$ it is isomorphic (as a module) to $\mathfrak{g}_2^\nu = \mathfrak{g}_{-1}^\nu$.

Remark 2.7. For any automorphism ρ of $\mathfrak{g}$ we have a natural $\mathbb{Z}$-graded Lie algebra isomorphism $\mathfrak{L}^\sigma \cong \mathfrak{L}^{\rho\sigma\rho^{-1}}$ given by $z^k x \longmapsto z^k\rho(x)$. Since the automorphism ν is defined by its order up to conjugation, this result implies that $\mathfrak{L}^\nu$ is also determined by the order of the automorphism ν. $\diamond$

A pair (α, k), where $\alpha \in \mathfrak{h}^\vee$ and $k \in \mathbb{Z}$, is called a *root* if the joint eigenspace

$$\mathfrak{g}_{(\alpha,k)}^\nu = \left\{x \in \mathfrak{g}_k^\nu \mid [h, x] = \alpha(h)x \;\forall h \in \mathfrak{h}\right\} \tag{2.21}$$

is non-zero. Let Φ be the set of all roots and Φ_k be the set of roots of the form (α, k). The triangular decomposition (2.20) of $\mathfrak{g}_0^\nu$ gives rise to the polarization $\Phi_0 = \Phi_0^- \cup \{(0, 0)\} \cup \Phi_0^+$. For convenience we introduce two more subsets of roots:

$$\begin{aligned} \Phi^+ &:= \Phi_0^+ \cup \{(\alpha, k) \in \Phi \mid k > 0\}, \\ \Phi^- &:= \Phi_0^- \cup \{(\alpha, k) \in \Phi \mid k < 0\}. \end{aligned} \tag{2.22}$$

The elements of Φ^+ and Φ^- are called *positive* and *negative roots* respectively. It is clear that $\Phi = \Phi^- \cup \{(0, 0)\} \cup \Phi^+$ and $-\Phi^+ = \Phi^-$. Denoting $z^k\mathfrak{g}_{(\alpha,k)}^\nu$ by $\mathfrak{L}_{(\alpha,k)}^\nu$ we get the root space decomposition

$$\mathfrak{L}^\nu = \bigoplus_{(\alpha,k)\in\Phi} \mathfrak{L}_{(\alpha,k)}^\nu, \tag{2.23}$$

where $\dim(\mathfrak{L}_{(\alpha,k)}^\nu) = 1$ if $\alpha \neq 0$ and $\mathfrak{L}_{(0,0)}^\nu = \mathfrak{h}$. The form B pairs the spaces $\mathfrak{L}_{(\alpha,k_1)}^\nu$ and $\mathfrak{L}_{(\beta,k_2)}^\nu$ non-degenerately if $(\alpha, k_1) + (\beta, k_2) = (0, 0)$; otherwise $B(\mathfrak{L}_{(\alpha,k_1)}^\nu, \mathfrak{L}_{(\beta,k_2)}^\nu) = 0$. Defining

$$\mathfrak{N}_\pm := \bigoplus_{(\alpha,k)\in\Phi^\pm} \mathfrak{L}_{(\alpha,k)}^\nu, \tag{2.24}$$

we obtain analogues of a triangular decomposition and Borel subalgebras for $\mathfrak{L}^\nu$, namely

$$\mathfrak{L}^\nu = \mathfrak{N}_- + \mathfrak{h} + \mathfrak{N}_+ \quad \text{and} \quad \mathfrak{B}_\pm := \mathfrak{h} + \mathfrak{N}_\pm. \tag{2.25}$$

Let $\{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots of $\mathfrak{g}_0^\nu$ with respect to (2.20) and α_0 be the corresponding minimal root. For any root α we write $\alpha^\vee$ for the unique element in $\mathfrak{h}$ such that $B(\alpha^\vee, -) = \alpha(-)$. The set

$$\Pi := \{\underbrace{(\alpha_0, 1)}_{=:\widetilde{\alpha}_0}, \underbrace{(\alpha_1, 0)}_{=:\widetilde{\alpha}_1}, \dots, \underbrace{(\alpha_n, 0)}_{=:\widetilde{\alpha}_n}\}. \tag{2.26}$$

is called *the simple root system* of $\mathfrak{L}^\nu$. It satisfies the following properties:

1. Any $(\alpha, k) \in \Phi$ can be uniquely written in the form $(\alpha, k) = \sum_{i=0}^n c_i \widetilde{\alpha}_i$, where $c_i \in \mathbb{Z}$. If the root (α, k) is positive (negative), then the coefficients c_i in its decomposition are all non-negative (non-positive);
2. The matrix $A := (a_{ij})$, where

$$a_{ij} := 2\frac{B(\alpha_i^\vee, \alpha_j^\vee)}{B(\alpha_j^\vee, \alpha_j^\vee)} \in \mathbb{Z} \qquad i, j \in \{0, 1, \dots, n\}, \tag{2.27}$$

is a generalized Cartan matrix of affine type. We call it *the affine matrix associated to* $\mathfrak{L}^\nu$. The Dynkin diagram corresponding to A is called *the Dynkin diagram of* $\mathfrak{L}^\nu$.

Let $\Lambda_0 := \{X_i^-, H_i, X_i^+\}_{i=1}^n$ be the set of standard Chevalley generators for $\mathfrak{g}_0^\nu$ with respect to the choice of simple roots we made earlier. Take two elements $X_0^\pm \in \mathfrak{L}_{(\pm\alpha_0, \pm 1)}$ such that

$$\left[X_0^+, X_0^-\right] = \frac{2\alpha_0^\vee}{B(\alpha_0^\vee, \alpha_0^\vee)} =: H_0. \tag{2.28}$$

By [20, Lemma X.5.8] the set $\Lambda := \Lambda_0 \cup \{X_0^-, H_0, X_0^+\}$ generates the whole Lie algebra $\mathfrak{L}^\nu$. For any $S \subsetneq \Pi$ we denote by $\mathfrak{S}^S$ the semi-simple subalgebra of $\mathfrak{L}^\nu$ generated by $\{X_i^-, H_i, X_i^+\}_{\widetilde{\alpha}_i \in S}$ with the induced triangular decomposition $\mathfrak{S}^S = \mathfrak{N}_-^S + \mathfrak{h}^S + \mathfrak{N}_+^S$. The subalgebras $\mathfrak{p}_\pm^S := \mathfrak{B}_\pm + \mathfrak{N}_\mp^S$ are the analogues for the parabolic subalgebras in the theory of semi-simple Lie algebras.

2.2.2. Classification of finite order automorphisms and regrading

We now explain the regrading procedure that makes it possible to transfer all the preceding results of this section to $\mathfrak{L}^\sigma$ for an arbitrary finite order automorphism σ. Let $s = (s_0, s_1, \dots, s_n)$ be a sequence of non-negative integers with at least one non-zero element. Using the properties of the simple root system (2.26) we can write

$$(0, |\nu|) = |\nu| \sum_{i=0}^n a_i \widetilde{\alpha}_i \tag{2.29}$$

for some unique positive integers a_i. We define a positive integer $m := |\nu| \sum_{i=0}^n a_i s_i$. The following results were proven in [20, Theorem X.5.15]:

1. The set $\{X_j^+(1)\}_{j=0}^n$ generates the Lie algebra $\mathfrak{g}$ and the relations

$$\sigma_{(s;|\nu|)}(X_j^+(1)) := e^{2\pi i s_j/m} X_j^+(1) \qquad 0 \le j \le n \tag{2.30}$$

define a unique automorphism $\sigma_{(s;|\nu|)}$ of $\mathfrak{g}$ of order m such that $\mathfrak{L}^\nu \cong \mathfrak{L}^{\sigma_{(s;|\nu|)}}$. In particular, $\nu = \sigma_{((1,0,\dots,0);|\nu|)}$;
2. Up to conjugation any finite order automorphism σ of $\mathfrak{g}$ arise in this way.

It follows immediately that for any finite order automorphism σ of $\mathfrak{g}$ there is an automorphism $\sigma_{(s;|\nu|)}$ and an outer automorphism ν of $\mathfrak{g}$ such that

$$\mathfrak{L}^\nu \underset{G^s}{\xrightarrow{\sim}} \mathfrak{L}^{\sigma_{(s;|\nu|)}} \xrightarrow{\sim} \mathfrak{L}^\sigma, \tag{2.31}$$

where the second isomorphism, given by conjugation, is described in Remark 2.7. The automorphism $\sigma_{(s;|\nu|)}$ is called *the automorphism of type* $(s; |\nu|)$. Note that the conjugacy class of the coset $\sigma_{(s;|\nu|)}\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ is represented by ν.

Now we describe the first isomorphism in the chain (2.31). Define *the s-height* $\mathrm{ht}_s(\alpha, k)$ of a root $(\alpha, k) \in \Phi$ in the following way: decompose (α, k) with respect to the simple root system Π, i.e. $(\alpha, k) = \sum_{i=0}^n c_i \widetilde{\alpha}_i$ and set

$$\mathrm{ht}_s(\alpha, k) := \sum_{i=0}^n c_i s_i. \tag{2.32}$$

We introduce a new $\mathbb{Z}$-grading on $\mathfrak{L}^\nu$, called $\mathbb{Z}$*-grading of type s*, by declaring $\deg(f) = 0$ for $f \in \mathfrak{h}$ and $\deg(f) = \mathrm{ht}_s(\alpha, k)$ for $f \in \mathfrak{L}^\nu_{(\alpha,k)}$. The isomorphism $G^s \colon \mathfrak{L}^\nu \longrightarrow \mathfrak{L}^{\sigma_{(s;|\nu|)}}$, called *regrading*, is given by

$$G^s(z^k x) := z^{\mathrm{ht}_s(\alpha,k)} x \qquad \forall z^k x \in \mathfrak{L}^\nu_{(\alpha,k)}. \tag{2.33}$$

If $\mathfrak{L}^{\nu}$ is equipped with the grading of type s and $\mathfrak{L}^{\sigma(s;|\nu|)}$ is equipped with the natural grading given by the powers of z, then G^s is a graded isomorphism. We write $G_s^{s'}$ for the resulting regrading $G^{s'} \circ (G^s)^{-1} \colon \mathfrak{L}^{\sigma(s;|\nu|)} \longrightarrow \mathfrak{L}^{\sigma(s';|\nu|)}$.

Remark 2.8. The grading given by $s = \mathbf{1} = (1, 1, \dots, 1)$ is called *the principle grading* and the corresponding automorphism $\sigma_{(\mathbf{1};|\nu|)}$ is *the Coxeter automorphism* of the pair $(\mathfrak{g}, \nu)$. $\diamond$

2.2.3. Structure theory (general case)

We finish the discussion of loop algebras by pushing the structure theory for $\mathfrak{L}^{\nu}$ to $\mathfrak{L}^{\sigma}$ through the chain of isomorphisms (2.31). We do it gradually, starting with the case $\sigma = \sigma_{(s;|\nu|)}$, $s = (s_0, s_1, \dots, s_n)$. Let Φ and Π, as before, be the set of all roots and the simple root system of $\mathfrak{L}^{\nu}$. From the definition of regrading it is clear that $G^s(\mathfrak{h}) = \mathfrak{h}$. This allows us to define the joint eigenspaces $\mathfrak{g}^{\sigma}_{(\alpha,\ell)}$, $\alpha \in \mathfrak{h}^{\vee}$, $\ell \in \mathbb{Z}$, using the exact same formula (2.21) and call (α, ℓ) a root of $\mathfrak{L}^{\sigma}$ if $\mathfrak{g}^{\sigma}_{(\alpha,\ell)} \neq 0$. Using regrading we can describe the root spaces $\mathfrak{L}^{\sigma}_{(\alpha,\ell)} := z^{\ell}\mathfrak{g}^{\sigma}_{(\alpha,\ell)}$ of $\mathfrak{L}^{\sigma}$ in terms of the root spaces of $\mathfrak{L}^{\nu}$, namely

$$G^s\left(\mathfrak{L}^{\nu}_{(\alpha,k)}\right) = \mathfrak{L}^{\sigma}_{(\alpha,\mathrm{ht}_s(\alpha,k))} \qquad \forall (\alpha,k) \in \Phi. \tag{2.34}$$

This gives a bijection between roots of $\mathfrak{L}^{\nu}$ and $\mathfrak{L}^{\sigma}$. More precisely, let Φ_{σ} be the set of all roots of $\mathfrak{L}^{\sigma}$, then

$$\Phi_{\sigma} = \{(\alpha, \mathrm{ht}_s(\alpha,k)) \mid (\alpha,k) \in \Phi\}. \tag{2.35}$$

The subset $\Pi^{\sigma} := \{(\alpha_0, s_0), (\alpha_1, s_1), \dots, (\alpha_n, s_n)\} \subseteq \Phi_{\sigma}$ is said to be the simple root system of $\mathfrak{L}^{\sigma}$. We again adopt the notation $\widetilde{\alpha}_i$ for the simple root (α_i, s_i). By definition of ht_s the root spaces $\mathfrak{L}^{\sigma}_{(\alpha,\ell_1)}$ and $\mathfrak{L}^{\sigma}_{(\beta,\ell_2)}$ are paired by the form B non-degenerately if $(\alpha,\ell_1)+(\beta,\ell_2) = (0,0)$; otherwise $B(\mathfrak{L}^{\sigma}_{(\alpha,\ell_1)}, \mathfrak{L}^{\sigma}_{(\beta,\ell_2)}) = 0$. It is evident from (2.34) that the subspaces $\mathfrak{N}_{\pm}, \mathfrak{B}_{\pm} \subseteq \mathfrak{L}^{\nu}$ are fixed under regrading and thus we can unambiguously use the same notations for them considered as subspaces of $\mathfrak{L}^{\sigma}$. Applying regrading to the set of generators $\Lambda = \{X_i^-, H_i, X_i^+\}_{i=0}^n$ of $\mathfrak{L}^{\nu}$ we obtain the set

$$\Lambda^{\sigma} := \left\{z^{-s_i}X_i^-(1), H_i, z^{s_i}X_i^+(1)\right\}_{i=0}^n \tag{2.36}$$

of generators of $\mathfrak{L}^{\sigma}$. When $S \subsetneq \Pi^{\sigma}$ we use the same notation $\mathfrak{S}^S$ to denote the semi-simple subalgebra of $\mathfrak{L}^{\sigma}$ generated by $\{z^{-s_i}X_i^-(1), H_i, z^{s_i}X_i^+(1)\}_{\widetilde{\alpha}_i \in S}$ with the induced triangular decomposition $\mathfrak{S}^S = \mathfrak{N}_-^S \dotplus \mathfrak{h}^S \dotplus \mathfrak{N}_+^S$. The corresponding parabolic subalgebras of $\mathfrak{L}^{\sigma}$ are defined using the same formulas, namely $\mathfrak{p}_{\pm}^S := \mathfrak{B}_{\pm} \dotplus \mathfrak{N}_{\mp}^S$. We also define

$$\mathfrak{n}_{\pm}^{\sigma} := \bigoplus_{(\alpha,0)\in\Phi_{\sigma}^{\pm}} \mathfrak{L}^{\sigma}_{(\alpha,0)} = \mathfrak{g}_0^{\sigma} \cap \mathfrak{N}_{\pm}. \tag{2.37}$$

This gives the triangular decomposition $\mathfrak{g}_0^{\sigma} = \mathfrak{n}_-^{\sigma} \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+^{\sigma}$.

Finally, we consider the case $\sigma = \rho\sigma_{(s;|\nu|)}\rho^{-1}$ for some $\rho \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. We denote the natural isomorphism

$$\mathfrak{L}^{\sigma(s;|\nu|)} \longrightarrow \mathfrak{L}^{\sigma}, \qquad z^k x \longmapsto z^k \rho(x) \tag{2.38}$$

with the same letter ρ. The roots of $\mathfrak{L}^{\sigma}$ with respect to the action of the Cartan subalgebra $\rho(\mathfrak{h})$ are of the form $(\alpha\rho^{-1}, \ell)$, where (α, ℓ) is a root of $\mathfrak{L}^{\sigma(s;|\nu|)}$, and the root spaces are described by

$$\mathfrak{L}^{\sigma}_{(\alpha\rho^{-1},\ell)} = \rho\left(\mathfrak{L}^{\sigma(s;|\nu|)}_{(\alpha,\ell)}\right). \tag{2.39}$$

The set of all roots is again denoted by Φ_{σ}, and its subset

$$\Pi^{\sigma} := \{(\alpha_0\rho^{-1}, s_0), (\alpha_1\rho^{-1}, s_1), \dots, (\alpha_n\rho^{-1}, s_n)\} \tag{2.40}$$

is called the simple root system of $\mathfrak{L}^{\sigma}$. Applying ρ to the generators (2.36) of $\mathfrak{L}^{\sigma(s;|\nu|)}$ we get the set

$$\Lambda^{\sigma} := \left\{z^{-s_i}\rho(X_i^-(1)), \rho(H_i), z^{s_i}\rho(X_i^+(1))\right\}_{i=0}^n \tag{2.41}$$

of generators of $\mathfrak{L}^{\sigma}$. Later, when there is no ambiguity, the same notations $X_i^{\pm}$ and H_i are used to denote the elements of generating sets (2.41) and (2.36). Combining (2.39) with (2.34) we define

$$\begin{aligned}
\mathfrak{n}_{\pm}^{\sigma} &:= \rho(\mathfrak{n}_{\pm}^{\sigma(s;|\nu|)}) = \bigoplus_{(\alpha,0)\in\Phi^{\pm}} \mathfrak{L}^{\sigma}_{(\alpha\rho^{-1},\mathrm{ht}_s(\alpha,k))},\\
\mathfrak{N}_{\pm}^{\sigma} &:= \rho(\mathfrak{N}_{\pm}) = \bigoplus_{(\alpha,k)\in\Phi^{\pm}} \mathfrak{L}^{\sigma}_{(\alpha\rho^{-1},\mathrm{ht}_s(\alpha,k))},\\
\mathfrak{B}_{\pm}^{\sigma} &:= \rho(\mathfrak{B}_{\pm}) = \mathfrak{N}_{\pm}^{\sigma} \dotplus \rho(\mathfrak{h}).
\end{aligned} \tag{2.42}$$

where Φ, as before, is the set of all roots of $\mathfrak{L}^{\nu}$. Note that this notation is in consistence with the one defined earlier.

Remark 2.9. Let $\sigma = \rho\sigma_{(s;|\nu|)}\rho^{-1}$ and A be the affine matrix associated to $\mathfrak{L}^{\sigma}$, defined in a way similar to (2.27). Then A coincides with the affine Cartan matrix of $\mathfrak{L}^{\nu}$ and so does the Dynkin diagram of $\mathfrak{L}^{\sigma}$. $\diamond$

2.2.4. Connection to Kac–Moody algebras

As the structure theory developed in the preceding subsections suggests, the notion of a loop algebra is closely related to the notion of an affine Kac–Moody algebra. More precisely, let A be an affine matrix of type $X_N^{(m)}$, $\mathfrak{g}$ be the simple finite-dimensional Lie algebra of type X_N and ν be an automorphism of $\mathfrak{g}$ induced by an automorphism of the corresponding Dynkin diagram with $|\nu| = m$. Then A is the Cartan matrix of $\mathfrak{L}^\nu$ and the affine Kac–Moody algebra $\mathfrak{K}(A)$ is isomorphic to

$$\mathfrak{L}^\nu \dotplus \mathbb{C}c \dotplus \mathbb{C}d, \tag{2.43}$$

where $\mathbb{C}c$ is the one-dimensional centre of $\mathfrak{K}(A)$, d is the additional derivation element that acts on $\mathfrak{L}^\nu$ as $z\frac{d}{dz}$ and the Lie bracket is described by

$$[z^k x, z^\ell y] = z^{k+\ell}[x, y] + kB(z^k x, z^\ell y)c \qquad \forall z^k x, z^\ell y \in \mathfrak{L}. \tag{2.44}$$

Consequently $\mathfrak{L}^\nu \cong [\mathfrak{K}(A), \mathfrak{K}(A)]/\mathbb{C}c$ and the form (2.18) on $\mathfrak{L}^\nu$ extends to a standard bilinear form on $\mathfrak{K}(A)$ in the sense of [23, §2].

3. The standard Lie bialgebra structure on $\mathfrak{L}^\sigma$ and its twists

Let $\mathfrak{K}(A)$ be a symmetrizable Kac–Moody algebra with a fixed invariant non-degenerate symmetric bilinear form B. Then it possesses a Lie bialgebra structure δ_0, called the standard Lie bialgebra structure on $\mathfrak{K}(A)$, given by

$$\delta_0(H_i) = 0, \qquad \delta_0(D_i) = 0, \qquad \delta_0(X_i^\pm) = \frac{B(\alpha_i^\vee, \alpha_i^\vee)}{2} H_i \wedge X_i^\pm, \tag{3.1}$$

where $\{X_i^-, H_i, X_i^+\} \cup \{D_i\}$ is a set of standard generators for $\mathfrak{K}(A)$ (see [12, Example 3.2] and [10, Example 1.3.8]). We can immediately see that δ_0 induces a Lie bialgebra structure on

$$[\mathfrak{K}(A), \mathfrak{K}(A)]/Z(\mathfrak{K}(A)), \tag{3.2}$$

where $Z(\mathfrak{K}(A))$ is the centre of $\mathfrak{K}(A)$. In particular, when A is an affine matrix and B is the form mentioned in Section 2.2.4 we get a Lie bialgebra structure δ_0^ν on $\mathfrak{L}^\nu$. Applying the methods described in Section 2.2 we induce a Lie bialgebra structure δ_0^σ, called *the standard Lie bialgebra structure*, on $\mathfrak{L}^\sigma$ for any finite order automorphism σ. Its twisted versions δ_t^σ are called *twisted standard structures*.

3.1. Pseudoquasitriangular structure

We want to prove that δ_t^σ is a pseudoquasitriangular Lie bialgebra structure, i.e. it is defined by an r-matrix. We restrict our attention to a special case $\sigma = \sigma_{(s;|\nu|)}$. The general result will then follow from the natural isomorphism mentioned in Remark 2.7.

Let C_k^σ be the projection of the Casimir element $C = \sum_{k=0}^{|\sigma|-1} C_k^\sigma \in \mathfrak{g} \otimes \mathfrak{g}$ on the eigenspace $\mathfrak{g}_k^\sigma \otimes \mathfrak{g}_{-k}^\sigma$. The triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_-^\sigma \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+^\sigma$ leads to the splitting $C_0^\sigma = C_-^\sigma + C_\mathfrak{h} + C_+^\sigma$, where $C_\pm^\sigma \in \mathfrak{n}_\pm^\sigma \otimes \mathfrak{n}_\mp^\sigma$ and $C_\mathfrak{h} \in \mathfrak{h} \otimes \mathfrak{h}$. We introduce a rational function $r_0^\sigma \colon \mathbb{C}^2 \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ defined by

$$r_0^\sigma(x, y) := \frac{C_\mathfrak{h}}{2} + C_-^\sigma + \frac{1}{(x/y)^{|\sigma|} - 1} \sum_{k=0}^{|\sigma|-1} \left(\frac{x}{y}\right)^k C_k^\sigma. \tag{3.3}$$

Remark 3.1. Formula (3.3) can be seen as a generalization of well-known r-matrices. Kulish introduced $r_0^{\sigma(\mathbf{1};1)}$ in [28]. More generally $r_0^{\sigma(\mathbf{1};|\nu|)}$ was introduced in [4] by Belavin and Drinfeld, which they later, in [5], called the simplest trigonometric solution. Jimbo used r_0^ν in [21] and the formula for r_0^{id} appears in the recent works [26,32] and [8] under the name "quasi-trigonometric r-matrix". ◇

The statement in [4, Lemma 6.22] suggests the following holomorphic relations between functions defined by (3.3).

Lemma 3.2. *Let $\sigma, \sigma' \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ be two automorphisms of types $(s; |\nu|)$ and $(s'; |\nu|)$ respectively, where $s = (s_0, s_1, \ldots, s_n)$ and $s' = (s_0', s_1', \ldots, s_n')$. Then*

1. *The equations $\alpha_i(\mu) = s_i'/|\sigma'| - s_i/|\sigma|$, $i \in \{0, 1, \ldots, n\}$, define a unique element $\mu \in \mathfrak{h}$ such that*

$$e^{u\,\mathrm{ad}(\mu)} f\left(e^{u/|\sigma|}\right) = \left(G_s^{s'} f\right)\left(e^{u/|\sigma'|}\right) \qquad \forall f \in \mathfrak{L}^\sigma,\ \forall u \in \mathbb{C}; \tag{3.4}$$

2. *The functions r_0^σ and $r_0^{\sigma'}$ satisfy the relation*

$$\left(e^{u\,\mathrm{ad}(\mu)} \otimes e^{v\,\mathrm{ad}(\mu)}\right) r_0^\sigma\left(e^{u/|\sigma|}, e^{v/|\sigma|}\right) = r_0^{\sigma'}\left(e^{u/|\sigma'|}, e^{v/|\sigma'|}\right) \qquad \forall u, v \in \mathbb{C}, u - v \notin 2\pi i\mathbb{Z}. \tag{3.5}$$

Proof. Using the formulas

$$|\sigma| = |\nu| \sum_{i=0}^{n} a_i s_i \quad \text{and} \quad |\sigma'| = |\nu| \sum_{i=0}^{n} a_i s_i', \tag{3.6}$$

we can easily deduce that the equations $\alpha_i(\mu) = s_i'/|\sigma'| - s_i/|\sigma|$ are consistent and define a unique element $\mu \in \mathfrak{h}$. Let $f = X_i^{\pm}$, $i \in \{0, 1, \ldots, n\}$. Then for all $u \in \mathbb{C}$ we have

$$\begin{aligned} e^{u\,\mathrm{ad}(\mu)} X_i^{\pm}\left(e^{u/|\sigma|}\right) &= e^{\pm u s_i/|\sigma|} \sum_{k\geq 0} \frac{u^k}{k!}\left(\pm\frac{s_i'}{|\sigma'|} \mp \frac{s_i}{|\sigma|}\right)^k X_i^{\pm}(1) \\ &= e^{\pm u s_i'/|\sigma'|} X_i^{\pm}(1) = \left(G_s^{s'} X_i^{\pm}\right)\left(e^{u/|\sigma'|}\right). \end{aligned} \tag{3.7}$$

Since $\mathfrak{L}^\sigma$ is generated by $X_i^{\pm}$, identity (3.7) proves the first statement. To verify the second statement we choose a basis $\{b^i_{(\alpha,k)}\}$ for each $\mathfrak{L}^\sigma_{(\alpha,k)}$ such that

$$B\left(b^i_{(\alpha,k)}, b^j_{(-\alpha,-k)}\right) = \delta_{ij} \qquad \forall (\alpha, k) \in \Phi_\sigma. \tag{3.8}$$

Setting $n_{(\alpha,k)} := \dim(\mathfrak{L}^\sigma_{(\alpha,k)})$ we can write

$$\left(\frac{y}{x}\right)^{-k+\ell|\sigma|} C_k^\sigma = \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} b^i_{(-\alpha,k-\ell|\sigma|)}(x) \otimes b^i_{(\alpha,-k+\ell|\sigma|)}(y) \qquad \forall x, y \in \mathbb{C}^*, \tag{3.9}$$

where Φ_σ^+ stands for the set of positive roots of $\mathfrak{L}^\sigma$. Then the Taylor series of r_0^σ in $y = 0$ for a fixed x is

$$r_0^\sigma(x, y) = \frac{C_{\mathfrak{h}}}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} b^i_{(-\alpha,-k)}(x) \otimes b^i_{(\alpha,k)}(y). \tag{3.10}$$

It converges absolutely in $|y| < |x|$ allowing us to perform the following calculation

$$\begin{aligned} \left(e^{u\,\mathrm{ad}(\mu)} \otimes e^{v\,\mathrm{ad}(\mu)}\right) r_0^\sigma\left(e^{u/|\sigma|}, e^{v/|\sigma|}\right) &= \frac{C_{\mathfrak{h}}}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} e^{u\,\mathrm{ad}(\mu)} b^i_{(-\alpha,-k)}\left(e^{u/|\sigma|}\right) \otimes e^{v\,\mathrm{ad}(\mu)} b^i_{(\alpha,k)}\left(e^{v/|\sigma|}\right) \\ &= \frac{C_{\mathfrak{h}}}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} \left(G_s^{s'} b^i_{(-\alpha,-k)}\right)\left(e^{u/|\sigma'|}\right) \otimes \left(G_s^{s'} b^i_{(\alpha,k)}\right)\left(e^{v/|\sigma'|}\right) \\ &= r_0^{\sigma'}\left(e^{u/|\sigma'|}, e^{v/|\sigma'|}\right), \end{aligned}$$

for $|e^{v/|\sigma|}| < |e^{u/|\sigma|}|$ or, equivalently, $|e^{v/|\sigma'|}| < |e^{u/|\sigma'|}|$. Equality (3.5) now follows by the identity theorem for holomorphic functions of several variables (see [17]). ■

Having this result at hand we can obtain the desired pseudoquasitriangularity for twisted standard structures δ_t^σ. Let us call a meromorphic function $r : \mathbb{C}^2 \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ *skew-symmetric* if $r(x, y) + \tau(r(y, x)) = 0$.

Theorem 3.3. *Let $\sigma \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ be a finite order automorphism and $t \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma$. Then $r_t^\sigma := r_0^\sigma + t$ is a skew-symmetric solution of the CYBE if and only if t is a classical twist of δ_0^σ. Moreover, if t is a classical twist of δ_0^σ, then the following relation holds*[4]*:*

$$\delta_t^\sigma(f)(x, y) = [f(x) \otimes 1 + 1 \otimes f(y), r_t^\sigma(x, y)] \qquad \forall f \in \mathfrak{L}^\sigma,\ \forall x, y \in \mathbb{C}^*. \tag{3.11}$$

Proof. First, assume that $\sigma = \sigma_{(\mathbf{1};|\nu|)}$ and $t = 0$. In this case [4, Proposition 6.1] implies that r_0^σ is a skew-symmetric solution of the CYBE and (3.11) follows immediately from comparing [4, Equations (6.4) and (6.5)] with the projections of the defining relations (3.1) to $\mathfrak{L}^\sigma$. Secondly, applying Lemma 3.2 we get the statement for an arbitrary finite order automorphism σ and $t = 0$. Finally, since r_0^σ is skew-symmetric, the skew-symmetry of t is equivalent to the skew-symmetry of r_t^σ and a straightforward computation gives the equality

$$\mathrm{CYB}(r_t^\sigma) = \mathrm{CYB}(r_0^\sigma) + \mathrm{CYB}(t) - \mathrm{Alt}((\delta_0^\sigma \otimes 1)t) = \mathrm{CYB}(t) - \mathrm{Alt}((\delta_0^\sigma \otimes 1)t), \tag{3.12}$$

which completes the proof. ■

[4] We define $(f \otimes g)(x, y) := f(x) \otimes g(y)$ for any $x, y \in \mathbb{C}^*$ and $f, g \in \mathfrak{L}^\sigma$.

We finish this subsection by relating r-matrices of the form r_t^σ to trigonometric r-matrices in the sense of the Belavin–Drinfeld classification [4].

Theorem 3.4. *Let t be a classical twist of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ and $r_t^\sigma = r_0^\sigma + t$ be the corresponding r-matrix. Then there exist a holomorphic function $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a trigonometric r-matrix $X\colon \mathbb{C} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ such that*

$$X(u-v) = (\varphi(u)^{-1}\otimes\varphi(v)^{-1}) r_t^\sigma(e^{u/|\sigma|}, e^{v/|\sigma|}). \tag{3.13}$$

Proof. Let

$$r(x,y) := r_t^\sigma(x,y) = \frac{1}{(x/y)^{|\sigma|}-1}\widetilde{C}\,(x/y) + g(x,y), \tag{3.14}$$

where $\widetilde{C}(z) := \sum_{k=0}^{|\sigma|-1} z^k C_k^\sigma$. Following the arguments in [3] and [26, Theorem 11.3] we rewrite the CYBE for r in the form

$$[r^{12}(x,y), r^{13}(x,z)] + [r^{12}(x,y)+r^{13}(x,z), g^{23}(y,z)] + \frac{1}{(y/z)^{|\sigma|}-1}[r^{12}(x,y)+r^{13}(x,z), \widetilde{C}^{23}(y/z)] = 0.$$

Calculating the limit $y \to z$ using L'Hospital's rule we obtain

$$[r^{12}(x,z), r^{13}(x,z)] + [r^{12}(x,z)+r^{13}(x,z), g^{23}(z,z) + |\sigma|^{-1}(\widetilde{C}'(1))^{23}] + \frac{z}{|\sigma|}[\partial_z r^{12}(x,z), \widetilde{C}^{23}(y/z)] = 0.$$

Applying the function $1\otimes L\colon \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$, where $L(a\otimes b) := [a,b]$, we get the equality

$$[r(x,z), r(x,z)] + [r(x,z), 1\otimes f(z)] + \frac{z}{|\sigma|}\partial_z r(x,z) = 0, \tag{3.15}$$

where $f(z) := L(g(z,z) + |\sigma|^{-1}\widetilde{C}'(1))$ and $[a\otimes b, c\otimes d] := [a,c]\otimes[b,d]$. Similarly, letting $x \to y$ in the CYBE for r and then applying $L\otimes 1$ we obtain the identity

$$[r(y,z), r(y,z)] - [r(y,z), f(y)\otimes 1] - \frac{y}{|\sigma|}\partial_y r(y,z) = 0. \tag{3.16}$$

Subtracting (3.15) from (3.16) and setting $x = y = e^{u/|\sigma|}$ and $z = e^{v/|\sigma|}$ we get

$$\partial_u r(e^{u/|\sigma|}, e^{v/|\sigma|}) + \partial_v r(e^{u/|\sigma|}, e^{v/|\sigma|}) = [h(u)\otimes 1 + 1\otimes h(v), r(e^{u/|\sigma|}, e^{v/|\sigma|})], \tag{3.17}$$

for $h(u) := f(e^{u/|\sigma|})$. Since h is holomorphic on $\mathbb{C}$, we can find a holomorphic function $\varphi\colon\mathbb{C}\longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that $\varphi'(z) = \mathrm{ad}(h(z))\varphi(z)$ and $\varphi(0) = \mathrm{id}_\mathfrak{g}$ (see [26, Proof of Theorem 11.3]). The connected component of $\mathrm{id}_\mathfrak{g}$ in the group $\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ is exactly the inner automorphisms of $\mathfrak{g}$ and thus $\varphi\colon\mathbb{C}\longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. Finally, the relation (3.17) implies that the r-matrix

$$\widetilde{X}(u,v) := (\varphi(u)^{-1}\otimes\varphi(v)^{-1}) r(e^{u/|\sigma|}, e^{v/|\sigma|}) \tag{3.18}$$

satisfies the equation $\partial_u\widetilde{X}(u,v) + \partial_v\widetilde{X}(u,v) = 0$. Therefore, we can define $X(u-v) := \widetilde{X}(u-v,0) = \widetilde{X}(u,v)$. The set of poles of X is $2\pi i\mathbb{Z}$ and hence it is a trigonometric r-matrix. ■

From now on r-matrices of the form $r_t^\sigma = r_0^\sigma + t$, where σ is a finite order automorphism of $\mathfrak{g}$ and t is a classical twists of δ_0^σ, are called *σ-trigonometric.*

3.2. Manin triple structure

The standard Lie bialgebra structure on an affine Kac–Moody algebra (3.1) can be defined using the standard Manin triple (see [12, Example 3.2] and [10, Example 1.3.8]). Restricting that triple to $\mathfrak{L}^\sigma$ we get a Manin triple defining the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. More precisely, δ_0^σ is defined by the Manin triple

$$(\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma, \Delta, W_0), \tag{3.19}$$

where Δ is the image of the diagonal embedding of $\mathfrak{L}^\sigma$ into $\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma$ and W_0 is defined by

$$W_0 := \left\{(f,g)\in\mathfrak{B}_+^\sigma\times\mathfrak{B}_-^\sigma \mid f+g\in\mathfrak{N}_+^\sigma+\mathfrak{N}_-^\sigma\right\}. \tag{3.20}$$

The form $\mathcal{B}$ on $\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma$ is given by

$$\mathcal{B}\left((f_1,f_2),(g_1,g_2)\right) := B(f_1,g_1) - B(f_2,g_2) \qquad \forall f_1,f_2,g_1,g_2\in\mathfrak{L}^\sigma, \tag{3.21}$$

where B is the form (2.18). From Theorem 2.4 we know that classical twists t of δ_0^σ are in one-to-one correspondence with Lagrangian subalgebras $W_t\subseteq\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma$ complementary to Δ and commensurable with W_0. We now describe the construction of such subalgebras using σ-trigonometric r-matrices.

Let $\psi \colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{C}-\mathrm{Vect}}(\mathfrak{g})$ and $\Psi \colon \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \longrightarrow \mathrm{End}_{\mathbb{C}-\mathrm{Vect}}(\mathfrak{L}^\sigma)$ be the natural maps given by $a \otimes b \longmapsto \kappa(b, -)a$ and $a \otimes b \longmapsto B(b, -)a$ respectively. Then we have the following useful identity

$$\mathrm{res}_{y=0}\left[\frac{1}{y}\psi(P(z,y))(f(y))\right] = \Psi(P)(f)(z) \qquad \forall P \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma,\ \forall f \in \mathfrak{L}^\sigma,\ \forall z \in \mathbb{C}^*. \tag{3.22}$$

Theorem 3.5. *Let t be a classical twist of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ and $r_t = r_0^\sigma + t$ be the corresponding σ-trigonometric r-matrix. Denote by $\pi_\mathfrak{h}$ and $\pi_\pm$ the projections of $\mathfrak{L}^\sigma$ onto $\mathfrak{h}$ and $\mathfrak{N}_\pm^\sigma$ respectively. Then the linear map $R_t := \pi_\mathfrak{h}/2 + \pi_- + \Psi(t)$ satisfies the relation*

$$\mathrm{res}_{y=0}\left[\frac{1}{y}\psi(r_t(z,y))(f(y))\right] = R_t(f)(z) \qquad \forall f \in \mathfrak{L}^\sigma,\ \forall z \in \mathbb{C}^*, \tag{3.23}$$

and the Lagrangian subalgebra W_t, corresponding to t, can be described in the following way

$$W_t = \{((R_t - 1)f, R_t f) \mid f \in \mathfrak{L}^\sigma\}. \tag{3.24}$$

Proof. We prove the theorem for $\sigma = \sigma_{(s;|\nu|)}$ and $t = 0$. The general result then follows by linearity and (3.22). Writing $r_0^\sigma(z,y)$ as series (3.10) and applying ψ we get

$$\psi(r_0^\sigma(z,y))(f(y)) = \frac{\psi(C_\mathfrak{h})(f(y))}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\le i\le n_{(\alpha,k)}}} \psi\left(b^i_{(-\alpha,-k)}(z) \otimes b^i_{(\alpha,k)}(y)\right)\Big(f(y)\Big). \tag{3.25}$$

The absolute convergence of the series in the annulus $\epsilon < |y| < |z|$ for any $\epsilon \in \mathbb{R}_+$ allows the componentwise calculation of the residue, i.e.

$$\mathrm{res}_{y=0}\left[\frac{1}{y}\psi(r_t(z,y))(f(y))\right] = \frac{\pi_\mathfrak{h}(f(y))}{2} + \sum_{(\alpha,k)\in\Phi_\sigma^+} \pi_{(-\alpha,-k)}(f(y)) = R_0(f)(z), \tag{3.26}$$

where $\pi_{(\alpha,k)}$ is the projection of $\mathfrak{L}^\sigma$ onto $\mathfrak{L}^\sigma_{(\alpha,k)}$.

For the second statement let us take an arbitrary $(w_1, w_2) \in W_0$. The relation $\pi_\mathfrak{h}(w_1) = -\pi_\mathfrak{h}(w_2)$ implies $(w_1, w_2) = ((R_0-1)(w_2-w_1), R_0(w_2-w_1))$. The desired result now follows from the fact that $(w_1, w_2) \longmapsto w_2 - w_1$ is an isomorphism between W_0 and $\mathfrak{L}^\sigma$. ■

Remark 3.6. For later sections it is convenient to define another, more geometric, Manin triple defining the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. Define $m := |\sigma|$, $O^\sigma := \mathbb{C}[z^m, z^{-m}]$ and $\widehat{O}^\sigma_\pm := \mathbb{C}(\!(z^{\pm m})\!)$.[5] The Lie algebra $\mathfrak{L}^\sigma$ is naturally an O^σ-module and hence we can extend it to $\widehat{\mathfrak{L}}^\sigma_\pm = \mathfrak{L}^\sigma \otimes_{O^\sigma} \widehat{O}^\sigma_\pm$. Equip the product Lie algebra $\widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-$ with the following bilinear form

$$\mathcal{B}((f_1,f_2),(g_1,g_2)) := \mathrm{res}_{z=0}\left[\frac{1}{z}\kappa(f_1,g_1)\right] - \mathrm{res}_{z=0}\left[\frac{1}{z}\kappa(f_2,g_2)\right], \tag{3.27}$$

where $\kappa(\sum_i a_i z^i, \sum_j b_j z^j) := \sum_{i,j}\kappa(a_i,b_j)z^{i+j}$ and $\mathrm{res}_{z=0}$ reads off the coefficient of z^{-1}. The restriction of this form to $\mathfrak{L}^\sigma \times \mathfrak{L}^\sigma$ is the form (3.21) defined earlier. Consider the subset

$$\widehat{W}_0 = \left\{(f,g) \in \widehat{\mathfrak{B}}^\sigma_+ \times \widehat{\mathfrak{B}}^\sigma_- \mid \widehat{\pi}^+_\mathfrak{h}(f) = -\widehat{\pi}^-_\mathfrak{h}(g)\right\} \subseteq \widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-, \tag{3.28}$$

where $\widehat{\mathfrak{B}}^\sigma_\pm := \mathfrak{h} \dotplus \widehat{\mathfrak{N}}^\sigma_\pm$, $\widehat{\mathfrak{N}}^\sigma_\pm$ stands for the completion of $\mathfrak{N}^\sigma_\pm$ with respect to the ideal $(z^{\pm m}) \subseteq \mathbb{C}[z^{\pm m}]$ and $\widehat{\pi}^\pm_\mathfrak{h} \colon \widehat{\mathfrak{L}}^\sigma_\pm \longrightarrow \mathfrak{h}$ are the canonical projections . Then $\widehat{W}_0$ is a Lagrangian subalgebra complementary to the diagonal embedding Δ of $\mathfrak{L}^\sigma$ into $\widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-$. Since the Lie bracket on W_0 is the restriction of the Lie bracket on $\widehat{W}_0$, the Manin triple

$$(\widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-, \Delta, \widehat{W}_0) \tag{3.29}$$

also defines the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$.

The geometric nature of this Manin triple is revealed in [1]: the sheaves used for construction of σ-trigonometric r-matrices can be viewed as formal gluing of twisted versions of $\widehat{W}_0$ with $\mathfrak{L}^\sigma \cong \Delta$ over the nodal Weierstraß cubic. ◇

3.3. Regular equivalence

Let us fix a finite order automorphism σ of $\mathfrak{g}$. We now turn to defining the notion of equivalence for twisted standard bialgebra structures on $\mathfrak{L}^\sigma$ which is compatible with the corresponding pseudoquasitriangular and Manin triple structures. In other words, we want equivalences of Lie bialgebras to induce equivalences of the corresponding Manin triples and

5 The notation $\mathbb{C}(\!(u)\!)$ is used to denote the ring of Laurent series of the form $\sum_{k=N}^\infty a_k u^k$, where $a_k \in \mathbb{C}$ and $N \in \mathbb{Z}$.

trigonometric r-matrices and vice versa. We stress that the notion of holomorphic equivalence used in the Belavin–Drinfeld classification [4] is unsuitable for our purpose, because in general it does not provide isomorphisms of loop algebras.

In the spirit of [33,34] we define a *regular equivalence* on the loop algebra $\mathfrak{L}^\sigma$ to be a regular function $\phi\colon \mathbb{C}^* \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ preserving the quasi-periodicity of $\mathfrak{L}^\sigma$, i.e.

$$\phi(\varepsilon_\sigma z) = \sigma\phi(z)\sigma^{-1}, \tag{3.30}$$

where $\varepsilon_\sigma = e^{2\pi i/|\sigma|}$. Recalling that $O^\sigma = \mathbb{C}[z^{|\sigma|}, z^{-|\sigma|}]$, we can equivalently define a regular equivalence on $\mathfrak{L}^\sigma$ to be an element of $\mathrm{Aut}_{O^\sigma-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$. The equivalence between these two definitions is given by $\phi(f)(z) := \phi(z)f(z)$

By definition (2.17) the space $\mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma$ can be viewed as the space of regular functions $T\colon \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ such that $(1\otimes\sigma)T(x,y) = T(x,\varepsilon_\sigma y)$ and $(\sigma\otimes 1)T(x,y) = T(\varepsilon_\sigma x, y)$. It is straightforward to check that if such a function T vanishes along the diagonal, i.e. $T(z,z) = 0$ for all $z \in \mathbb{C}^*$, then it is divisible by $(x/y)^{|\sigma|} - 1$. Applying this observation to the function

$$(\phi(x)\otimes\phi(y)) \sum_{k=0}^{|\sigma|-1} \left(\frac{x}{y}\right)^k C_k^\sigma - \sum_{k=0}^{|\sigma|-1} \left(\frac{x}{y}\right)^k C_k^\sigma, \tag{3.31}$$

where $\phi \in \mathrm{Aut}_{O^\sigma-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$, we see that $(\phi(x)\otimes\phi(y))r_t^\sigma(x,y) = r_0^\sigma(x,y) + s(x,y)$ for some classical twist s, i.e. it is again a σ-trigonometric r-matrix.

The following theorem demonstrates that the notion of a regular equivalence meets all our needs.

Theorem 3.7. *Let ϕ be a regular equivalence on $\mathfrak{L}^\sigma$ and $s, t \in \mathfrak{L}^\sigma\otimes\mathfrak{L}^\sigma$ be two classical twists of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. The following are equivalent:*

1. *$r_t^\sigma(x,y) = (\phi(x)\otimes\phi(y))r_s^\sigma(x,y)$ for all $x, y \in \mathbb{C}^*$, $x^{|\sigma|} \neq y^{|\sigma|}$;*
2. *$\delta_t^\sigma\phi = (\phi\otimes\phi)\delta_s^\sigma$;*
3. *$W_t = (\phi\times\phi)W_s$.*

Proof. "1. $\implies$ 3.": If $r_t(x,y) = (\phi(x)\otimes\phi(y))r_s(x,y)$ for all $x,y\in\mathbb{C}^*$, $x^{|\sigma|}\neq y^{|\sigma|}$, then (3.23) implies $R_t = \phi R_s \phi^*$. Since the adjoint of $\phi(z)$ with respect to the Killing form is $\phi(z)^{-1}$, we have $\phi^* = \phi^{-1}$. The formula (3.24) applied to both W_t and W_s gives

$$(\phi\times\phi)W_s = \left\{\left(\phi(R_s-1)\phi^{-1}(\phi f), \phi R_s\phi^{-1}(\phi f)\right) \mid f \in \mathfrak{L}^\sigma\right\} = W_t. \tag{3.32}$$

"3. $\implies$ 2.": Assuming $W_t = (\phi\times\phi)W_s$, we can easily see that $\phi\times\phi$ is an isomorphism of Manin triples $(\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma, \Delta, W_t)$ and $(\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma, \Delta, W_s)$. Identifying Δ with $\mathfrak{L}^\sigma$ and applying Remark 2.2 we immediately get the desired isomorphism $\phi\colon(\mathfrak{L}^\sigma,\delta_t)\longrightarrow(\mathfrak{L}^\sigma,\delta_s)$.

"2. $\implies$ 1.": Since $\mathfrak{L}^\sigma$ has no non-trivial finite-dimensional ideals (see [23, Lemma 8.6]), the only element in $\mathfrak{L}^\sigma\otimes\mathfrak{L}^\sigma$ invariant under the adjoint action of $\mathfrak{L}^\sigma$ is 0.

Applying this result to the equality

$$\begin{aligned}
[\phi(f)(x)\otimes 1 + 1\otimes\phi(f)(y), r_t(x,y)] &= \delta_t\phi(f)(x,y) = (\phi(x)\otimes\phi(y))\delta_s(f)(x,y) \\
&= (\phi(x)\otimes\phi(y))\,[f(x)\otimes 1 + 1\otimes f(y), r_s(x,y)] \\
&= [\phi(f)(x)\otimes 1 + 1\otimes\phi(f)(y), (\phi(x)\otimes\phi(y))r_s(x,y)]\,,
\end{aligned} \tag{3.33}$$

where $f\in\mathfrak{L}^\sigma$ and $x,y\in\mathbb{C}^*$, $x^{|\sigma|}\neq y^{|\sigma|}$, we get the last implication. ■

We say that two twisted standard bialgebra structures or σ-trigonometric r-matrices are *regularly equivalent* if one of the equivalent conditions in Theorem 3.7 holds.

4. The main classification theorem and its consequences

Before stating the main classification theorem we recall the notion of a Belavin–Drinfeld quadruple for an arbitrary finite order automorphism σ, defined in [4], and then associate it with a classical twist of the standard Lie bialgebra structure δ_0^σ.

We start with the case $\sigma = \sigma_{(s;|\nu|)}$. Let Π^σ, Λ^σ and Φ_σ be as at the end of Section 2.2. A Belavin–Drinfeld (BD) quadruple is a quadruple $Q = \left(\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h}\right)$, where Γ_1 and Γ_2 are proper subsets of the simple root system Π^σ, $\gamma\colon\Gamma_1\longrightarrow\Gamma_2$ is a bijection and $t_\mathfrak{h}\in\mathfrak{h}\wedge\mathfrak{h}$ such that

1. $B\left(\alpha_{\gamma(i)}^\vee, \alpha_{\gamma(j)}^\vee\right) = B\left(\alpha_i^\vee, \alpha_j^\vee\right)$ for all $\widetilde{\alpha}_i, \widetilde{\alpha}_j \in \Gamma_1$, where $\widetilde{\alpha}_{\gamma(i)} := \gamma(\widetilde{\alpha}_i)$;
2. For any $\widetilde{\alpha}_i \in \Gamma_1$ there is a positive integer k such that $\gamma^k(\widetilde{\alpha}_i) \notin \Gamma_1$;

3. $(\alpha_{\gamma(i)} \otimes 1 + 1 \otimes \alpha_i)(t_{\mathfrak{h}} + C_{\mathfrak{h}}/2) = 0$ for all $\widetilde{\alpha}_i \in \Gamma_1$.

The bijection γ induces an isomorphism $\theta_\gamma : \mathfrak{S}^{\Gamma_1} \longrightarrow \mathfrak{S}^{\Gamma_2}$, $\theta_\gamma(z^{\pm s_i} X_i^{\pm}(1)) := z^{\pm s_{\gamma(i)}} X_{\gamma(i)}^{\pm}(1)$, which we extend by 0 to the whole $\mathfrak{L}^\sigma$. Let $\Phi_1 \subseteq \Phi_\sigma$ be the subset of roots that can be written as linear combinations of elements in Γ_1. For each $\widetilde{\alpha} \in \Phi_1$ we choose an element $b_{\widetilde{\alpha}} \in \mathfrak{L}^\sigma_{\widetilde{\alpha}}$ such that $B(b_{\widetilde{\alpha}}, b_{-\widetilde{\alpha}}) = 1$ and construct the following skew-symmetric tensor

$$t_Q^\sigma := t_{\mathfrak{h}} + \sum_{\widetilde{\alpha} \in \Phi_1^+} \sum_{j=1}^{\infty} b_{-\widetilde{\alpha}} \wedge \theta_\gamma^j (b_{\widetilde{\alpha}}) \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma, \tag{4.1}$$

where $\Phi_1^+ = \Phi_1 \cap \Phi_\sigma^+$ and the second sum has only finitely many non-zero terms since θ_γ is nilpotent by condition 2.

We write $r_Q^\sigma, \delta_Q^\sigma, R_Q$ and W_Q instead of $r_{t_Q^\sigma}^\sigma, \delta_{t_Q^\sigma}^\sigma, R_{t_Q^\sigma}$ and $W_{t_Q^\sigma}$ respectively. In the case $s = (1, \dots, 1)$ the functions r_Q^σ and R_Q as well as the Cayley transform of R_Q were studied in detail in [4]. Using regrading and Lemma 3.2 we derive the following statements:

- r_t^σ is a skew-symmetric solution of CYBE. Hence Theorem 3.3 implies that t_Q^σ is a classical twist of δ_0^σ;
- The inhomogeneous system of linear equations constraining $t_{\mathfrak{h}}$ is consistent. The dimension of its solution space is $\ell(\ell-1)/2$, where $\ell = |\Pi^\sigma \setminus \Gamma_1|$;
- Setting $\theta^{\pm}_{\gamma^{\pm 1}} := \theta_{\gamma^{\pm 1}}|_{\mathfrak{N}_\pm}$, we have

$$R_Q = \theta_\gamma^+ (\theta_\gamma^+ - \pi_+)^{-1} + (\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2) + (\pi_- - \theta^-_{\gamma^{-1}})^{-1}; \tag{4.2}$$

- Let $\mathfrak{h}_1 := \mathrm{im}(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)$ and $\mathfrak{h}_2 := \mathrm{im}(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)$. The Cayley transform of R_Q is the triple (C_Q^1, C_Q^2, θ_Q), where

$$\begin{aligned} C_Q^1 &:= \mathrm{im}(R_Q - \mathrm{id}) = \mathfrak{N}_+ \dotplus \mathfrak{h}_1 \dotplus \mathfrak{N}_-^{\Gamma_1}, \\ C_Q^2 &:= \mathrm{im}(R_Q) = \mathfrak{N}_+^{\Gamma_2} \dotplus \mathfrak{h}_2 \dotplus \mathfrak{N}_-, \end{aligned} \tag{4.3}$$

and θ_Q is the unique gluing of θ_γ with the natural isomorphism

$$\begin{aligned} \phi \colon \frac{\mathrm{im}(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)}{\ker(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)} &\longrightarrow \frac{\mathrm{im}(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)}{\ker(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)}, \\ [(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)(h)] &\longmapsto [(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)(h)], \end{aligned} \tag{4.4}$$

which coincides with θ_γ on the intersection of the domains. The subalgebra W_Q is then given by

$$W_Q = \left\{ (x, y) \in C_Q^1 \times C_Q^2 \mid \theta_Q([x]) = [y] \right\}. \tag{4.5}$$

Conjugating σ by $\rho \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ we extend all statements and constructions given above to an arbitrary finite order automorphism of $\mathfrak{g}$.

Theorem 4.1 (*The Main Classification Theorem*). *For any classical twist t of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ there is a regular equivalence ϕ of $\mathfrak{L}^\sigma$ and a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_{\mathfrak{h}})$ such that*

$$\delta_t^\sigma \phi = (\phi \otimes \phi) \delta_Q^\sigma. \tag{4.6}$$

Furthermore, if $Q' = (\Gamma_1', \Gamma_2', \gamma', t_{\mathfrak{h}}')$ is another BD quadruple, the twisted bialgebra structures δ_Q^σ and $\delta_{Q'}^\sigma$ are regularly equivalent if and only if there is an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(\Gamma_i) = \Gamma_i'$ for $i = 1, 2$, $\vartheta \gamma \vartheta^{-1} = \gamma'$ and $(\vartheta \otimes \vartheta) t_{\mathfrak{h}} = t_{\mathfrak{h}}'$, which we denote by $\vartheta(Q) = Q'$.

We put off the proof of the theorem to Section 5. The rest of this section is devoted to various consequences of Theorem 4.1 and to the proof of its first part in the special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\sigma = \mathrm{id}$.

Example 4.2 (*Quasi-constant r-matrices*). *Let $r^{\mathrm{const}} \in \mathfrak{g} \otimes \mathfrak{g}$ be a constant r-matrix satisfying the condition*

$$r^{\mathrm{const}} + \tau(r^{\mathrm{const}}) = C. \tag{4.7}$$

Then the meromorphic function

$$r(x, y) := \frac{C}{(x/y) - 1} + r^{\mathrm{const}} \tag{4.8}$$

is a skew-symmetric solution of CYBE, called quasi-constant *r-matrix. The converse is also true: if (4.8) is a skew-symmetric r-matrix, then r^{const} is a constant r-matrix satisfying (4.7).*

It was shown in [5, Theorem 6.1] that any constant r-matrix satisfying (4.7) can be transformed by a suitable automorphism of $\mathfrak{g}$ into another r-matrix r_Q^{const} which satisfies the same condition (4.7) and which is completely determined by a constant

BD quadruple Q. That means $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$, where Γ_1 and Γ_2 are subsets of simple roots of $\mathfrak{g}$. Identifying simple roots $\alpha_1, \dots, \alpha_n$ of $\mathfrak{g}$ with simple roots $(\alpha_1, 0), \dots, (\alpha_n, 0)$ of $\mathfrak{L}$ we make a constant BD quadruple Q into a BD quadruple described at the beginning of this section. Comparing the formulas (4.1) and [5, Equation 6.8] we see that

$$r_Q^{\mathrm{id}}(x,y) = \frac{C}{(x/y)-1} + r_Q^{\mathrm{const}}. \quad \diamond$$

Remark 4.3. In Section 5 we actually prove an even stronger version of Theorem 4.1. More precisely, we show that if two σ-trigonometric r-matrices r_Q^σ and $r_{Q'}^\sigma$, given by BD quadruples Q and Q', are locally holomorphically equivalent, i.e.

$$(\varphi(x) \otimes \varphi(y)) r_Q^\sigma(x,y) = r_{Q'}^\sigma$$

for some holomorphic function $\varphi\colon U \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ defined in a neighbourhood U of 0, then these r-matrices are automatically regularly equivalent and there is a Dynkin diagram automorphism ϑ such that $\vartheta(Q) = Q'$ (see Theorem 5.10). This fact was stated without a proof in [4, Section 6.4, Remark 4], and we are not aware of any other proof. $\diamond$

4.1. Classification of twists for parabolic subalgebras

To simplify the notation we again assume $\sigma = \sigma_{(s;|\nu|)}$. The following results can be stated for an arbitrary finite order automorphism by applying conjugation.

Let $S \subsetneq \Lambda^\sigma$ be a proper subset of standard generators of $\mathfrak{L}^\sigma$. It is easy to see that the standard Lie bialgebra structure δ_0^σ restricts to both $\mathfrak{S}^S$ and $\mathfrak{p}_\pm^S$. Such induced Lie bialgebra structures can be defined using modifications of the Manin triple (3.19). For example, the Lie bialgebra structure $(\mathfrak{p}_+^S, \delta_0^\sigma|_{\mathfrak{p}_+^S})$ is defined by the Manin triple

$$\left((\mathfrak{S}^S + \mathfrak{h}) \times \mathfrak{L}^\sigma, \Delta^S, W_0^S\right), \tag{4.9}$$

where $W_0^S = W_0 \cap ((\mathfrak{S}^S + \mathfrak{h}) \times \mathfrak{L}^\sigma)$ and $\Delta^S = \{(\pi_S(f), f) \mid f \in \mathfrak{p}_+^S\}$ for the canonical projection $\pi_S\colon \mathfrak{p}_+^S \longrightarrow (\mathfrak{S}^S + \mathfrak{h}) = \mathfrak{p}_+^S/(\mathfrak{p}_+^S)^\perp$. The following theorem gives a classification of classical twists of the restricted Lie bialgebra structure $\delta_0^\sigma|_{\mathfrak{p}_+^S}$ or, equivalently, classical twists of δ_0^σ contained in $\mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$.

Theorem 4.4 (*The Classification Theorem for Parabolic Subalgebras*)**.** *For any classical twist $t \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$ of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ there exists a regular equivalence ϕ that restricts to an automorphism of $\mathfrak{p}_+^S$ and a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ such that*

$$\Gamma_1 \subseteq S \quad \text{and} \quad \delta_t^\sigma \phi = (\phi \otimes \phi)\delta_Q^\sigma. \tag{4.10}$$

Let $Q' = (\Gamma_1', \Gamma_2', \gamma', t_\mathfrak{h}')$, $\Gamma_1' \subseteq S$, be another BD quadruple. A regular equivalence between twisted standard structures δ_Q^σ and $\delta_{Q'}^\sigma$ restricts to an automorphism of $\mathfrak{p}_+^S$ if and only if the induced Dynkin diagram automorphism ϑ preserves S, i.e. $\vartheta(S) = S$.

Remark 4.5. For a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ with $\Gamma_1 \subseteq S$, formula (4.1) directly implies that $t_Q \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$. In particular t_Q is a twist of $\delta_0^\sigma|_{\mathfrak{p}_+^S}$. $\diamond$

The proof of Theorem 4.4 is based on the following three structural results for $\mathfrak{L}^\sigma$.

Lemma 4.6.

1. *A subalgebra $\mathfrak{a}$ of $\mathfrak{L}^\sigma$ containing a coisotropic subalgebra $\mathfrak{h}_1$ of $\mathfrak{h}$ satisfies $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{a}$;*
2. *A subalgebra $\mathfrak{p}$ of $\mathfrak{L}^\sigma$ containing $\mathfrak{B}_\pm$ is of the form $\mathfrak{p}_\pm^{S'}$ for some $S' \subseteq \Pi^\sigma$;*
3. *A mapping $\phi \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$ fixing $\mathfrak{B}_+$ or $\mathfrak{B}_-$ induces an automorphism of the Dynkin diagram of $\mathfrak{L}^\sigma$.*

Proof. 1.: We can write

$$\mathfrak{L}^\sigma = \bigoplus_{\alpha' \in \mathfrak{h}_1^\vee} \mathfrak{L}_{\alpha'}^\sigma = \bigoplus_{\alpha' \in \mathfrak{h}_1^\vee} \bigoplus_{\substack{\alpha \in \mathfrak{h}^\vee \\ \alpha|_{\mathfrak{h}_1} = \alpha'}} \mathfrak{L}_\alpha^\sigma, \tag{4.11}$$

where $\mathfrak{L}_\alpha^\sigma = \{f \in \mathfrak{L}^\sigma \mid [h,f] = \alpha(h)f \ \forall h \in \mathfrak{h}\} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{L}_{(\alpha,k)}^\sigma$ for any $\alpha \in \mathfrak{h}^\vee$ and similarly $\mathfrak{L}_{\alpha'}^\sigma = \{f \in \mathfrak{L}^\sigma \mid [h,f] = \alpha'(h)f \ \forall h \in \mathfrak{h}_1\}$ for any $\alpha' \in \mathfrak{h}_1^\vee$. Assume there are distinct $\alpha_1, \alpha_2 \in \mathfrak{h}^\vee$ such that $\mathfrak{L}_{\alpha_1}^\sigma, \mathfrak{L}_{\alpha_2}^\sigma \neq 0$ and $(\alpha_1 - \alpha_2)|_{\mathfrak{h}_1} = 0$. Since $\mathfrak{h}_1$ is coisotropic inside $\mathfrak{h}$, we have $(\alpha_1 - \alpha_2)^\vee \in \mathfrak{h}_1^\perp \subseteq \mathfrak{h}_1$. From [20, Lemma X.5.6] it follows that $\alpha_1 - \alpha_2 = 0$ which, in its turn, implies that for any $\alpha' \in \mathfrak{h}_1^\vee$ there exists a unique weight $\alpha \in \mathfrak{h}^\vee$ such that $\mathfrak{L}_{\alpha'}^\sigma = \mathfrak{L}_\alpha^\sigma$. This observation combined with $[\mathfrak{h}_1, \mathfrak{a}] \subseteq \mathfrak{a}$ allows us to write (see [23, Proposition 1.5])

$$\mathfrak{a} = \bigoplus_{\alpha' \in \mathfrak{h}_1^\vee} \mathfrak{L}_{\alpha'}^\sigma \cap \mathfrak{a} = \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{L}_\alpha^\sigma \cap \mathfrak{a}, \tag{4.12}$$

implying $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{a}$.

2.: Without loss of generality assume that $\mathfrak{p}$ contains $\mathfrak{B}_+$. The inclusion $\mathfrak{h} \subseteq \mathfrak{p}$ and [23, Proposition 1.5] imply that

$$\mathfrak{p} = \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{L}^\sigma_\alpha \cap \mathfrak{p}. \tag{4.13}$$

Take $X \in \mathfrak{L}^\sigma_{-\alpha} \cap \mathfrak{p} \cap \mathfrak{N}_-$ for some $\alpha \neq 0$. Let j be the maximal non-negative integer such that the $\mathfrak{L}_{(-\alpha,-j)}$-component of X is non-zero. The structure theory of $\mathfrak{L}^\sigma$ implies $\dim(\mathfrak{L}_{(-\alpha,-j)}) = 1$. Assume that $(-\alpha, -j+k)$ is a root for some positive integer k. Decomposing the difference of $(-\alpha, -j)$ and $(-\alpha, -j+k)$ into the sum of simple roots we get a relation of the form $\sum_{i=0}^n c_i\widetilde{\alpha}_i = (0, k)$. Then the identity $\sum_{i=0}^n c_i\alpha_i = 0$ and (2.29) imply that k is an integer multiple of $\sum_{i=0}^n a_i s_i$. Using [23, Theorem 5.6.b] we see that $(0, k)$ is a root. Applying [20, Lemma X.5.5'.(iii)] iteratively we see that

$$\bigoplus_{k \geq 0} \mathfrak{L}_{(-\alpha,-j+k)} \subseteq \mathfrak{p}. \tag{4.14}$$

Following the proof of [24, Lemma 1.5] we now show that $\mathfrak{p} = \mathfrak{p}^{S'}_+$, where

$$S' = \{(\alpha_i, s_i) \in \Pi^\sigma \mid \mathfrak{L}_{(-\alpha_i,-s_i)} \subseteq \mathfrak{p}\}. \tag{4.15}$$

Assume the claim is false. Let $(-\gamma, -\ell) \notin \operatorname{span}_\mathbb{Z}(S')$ be a negative root of maximal height such that there exists an element $Y \in \mathfrak{L}^\sigma_{-\gamma} \cap \mathfrak{p} \cap \mathfrak{N}_-$ with a non-zero $\mathfrak{L}_{(-\gamma,-\ell)}$-component $Y_{-\ell}$. Then there exists $(\alpha_j, s_j) \in \Pi^\sigma \setminus S'$ such that $[X_j^+, Y_{-\ell}] \neq 0$ and $(-\gamma + \alpha_j, -\ell + s_j) \in \operatorname{span}_\mathbb{Z}(S')$, where X_j^+ is the standard generator of $\mathfrak{L}^\sigma$. Note that (4.14) implies $\gamma \neq \alpha_j$. By the structure theory of loop algebras we can find $Z \in \mathfrak{L}^\sigma_{(\gamma-\alpha_j,\ell-s_j)} \subseteq \mathfrak{B}_+ \subseteq \mathfrak{p}$ such that

$$B([X_j^+, Y_{-\ell}], Z) \neq 0. \tag{4.16}$$

The invariance of the form B then gives $0 \neq [Y_{-\ell}, Z] \in \mathfrak{L}^\sigma_{(-\alpha_j,-s_j)}$. Applying formula (4.14) to $X = [Y, Z] \in \mathfrak{p}$ we get $(\alpha_j, s_j) \in S'$ contradicting our choice of (α_j, s_j).

3.: Assume that $\phi(\mathfrak{B}_+) = \mathfrak{B}_+$. Since $\mathfrak{N}_+ = [\mathfrak{B}_+, \mathfrak{B}_+]$, we see that ϕ also fixes $\mathfrak{N}_+$ and $\mathfrak{h}$. By [24, Lemma 1.29] the automorphism ϕ maps Π^σ to a root basis. But since ϕ fixes $\mathfrak{N}_+$, this root basis consists of positive roots and the only root basis in the set of positive roots is Π^σ. Hence $\phi(\Pi^\sigma) = \Pi^\sigma$ and thus ϕ induces an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$. ■

Proof of Theorem 4.4. We prove the statement for $\sigma = \sigma_{(s;|\nu|)}$. The general result it obtained using conjugation. By Theorem 4.1 there is a regular equivalence ϕ_1 on $\mathfrak{L}^\sigma$ and a BD quadruple $Q' = (\Gamma_1', \Gamma_2', \gamma', t_\mathfrak{h}')$ such that $(\phi_1 \times \phi_1)W_t = W_{Q'}$. Since $t \in \mathfrak{p}^S_+ \otimes \mathfrak{p}^S_+$ we have $W_t \subseteq \mathfrak{p}^S_+ \times \mathfrak{L}^\sigma$. Let $\mathfrak{h}_1 \subseteq \mathfrak{h}$ be the image of $\psi(t_\mathfrak{h}) - \mathrm{id}_\mathfrak{h}/2$. Since $t_\mathfrak{h}$ is skew-symmetric, this is easily seen to be a coisotropic subspace of $\mathfrak{h}$. Then

$$C^1_Q = \mathfrak{N}_+ \dotplus \mathfrak{h}_1 \dotplus \mathfrak{N}_-^{\Gamma_1} \subseteq \phi_1(\mathfrak{p}^S_+) \tag{4.17}$$

and, in particular, we have the inclusion $\mathfrak{h}_1 \subseteq \phi_1(\mathfrak{p}^S_+)$. By the first part of Lemma 4.6 we have

$$[\mathfrak{h}, \phi_1(\mathfrak{p}^S_+)] \subseteq \phi_1(\mathfrak{p}^S_+). \tag{4.18}$$

Since $\mathfrak{p}^S_+$ is self-normalizing, $\phi_1(\mathfrak{p}^S_+)$ is self-normalizing as well. Therefore we get $\mathfrak{h} \subseteq \phi_1(\mathfrak{p}^S_+)$ and consequently $\mathfrak{B}_+ \subseteq \phi_1(\mathfrak{p}^S_+)$. Then the second statement of Lemma 4.6 shows that $\phi_1(\mathfrak{p}^S_+) = \mathfrak{p}^{S'}_+$ for some $S' \subsetneq \Pi^\sigma$. The inclusion (4.17) implies that $\Gamma_1 \subseteq S'$.

Define $\mathfrak{B}' := \phi_1^{-1}(\mathfrak{B}_+)$. The subalgebra $\mathfrak{B}'/\mathfrak{p}^{S,\perp}_+$, being the preimage of the Borel subalgebra $\mathfrak{B}_+/\mathfrak{p}^{S',\perp}_+$ of $\mathfrak{S}^{S'} + \mathfrak{h} = \mathfrak{p}^{S'}_+/\mathfrak{p}^{S',\perp}_+$ under ϕ_1, is a Borel subalgebra of $\mathfrak{S}^S + \mathfrak{h} = \mathfrak{p}^S_+/\mathfrak{p}^{S,\perp}_+$. Therefore, by the conjugacy theorem for Borel subalgebras, there exists an inner automorphism ϕ_2 of $\mathfrak{S}^S + \mathfrak{h}$ mapping $\mathfrak{B}'/\mathfrak{p}^{S,\perp}_+$ to $\mathfrak{B}_+/\mathfrak{p}^{S,\perp}_+$. It can be seen from [20, Lemma X.5.5] that ad_x is nilpotent on $\mathfrak{L}^\sigma$ for any $x \in \mathfrak{L}^\sigma_{(\alpha,k)}$ and $\alpha \neq 0$. Combining this result with the equality

$$\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{S}^S + \mathfrak{h}) = \langle e^{\mathrm{ad}_x} \mid x \in \mathfrak{L}^\sigma_{(\alpha,k)}, (\alpha, k) \in \Phi_\sigma \cap \operatorname{span}_\mathbb{Z}(S), \alpha \neq 0\rangle$$

(see [6, §3.2]), we can view ϕ_2 as a regular equivalence on $\mathfrak{L}^\sigma$ that restricts to an automorphism of $\mathfrak{p}^S_+$ and maps $\mathfrak{B}'$ to $\mathfrak{B}_+$. The composition $\phi_2\phi_1^{-1}$ is then an automorphism of $\mathfrak{L}^\sigma$ mapping $\mathfrak{p}^{S'}_+$ to $\mathfrak{p}^S_+$ and fixing the Borel subalgebra $\mathfrak{B}_+$. The third part of Lemma 4.6 implies that $\phi_2\phi_1^{-1}$ induces an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(S') = S$. Applying the second part of Theorem 4.1 to ϑ we obtain a regular equivalence ϕ_3 such that $(\phi_3 \times \phi_3)W_{Q'} = W_{Q:=\vartheta(Q')}$. The composition $\phi := \phi_3\phi_1$ and the quadruple Q satisfy all the requirements of the theorem. ■

4.2. Quasi-trigonometric solutions of CYBE

Letting $\sigma = \mathrm{id}$ and $S = \Pi \setminus \{(\alpha_0, 1)\}$ the corresponding parabolic subalgebra $\mathfrak{p}^S_+$ becomes $\mathfrak{g}[z]$. The solutions to CYBE of the form $r_t = r_0 + t$, where $t \in \mathfrak{g}[z]^{\otimes 2}$, are called *quasi-trigonometric*. Two quasi-trigonometric solutions r_t and r_s are called *polynomially equivalent* if there exists a $\phi \in \mathrm{Aut}_{\mathbb{C}[z]-\mathrm{LieAlg}}(\mathfrak{g}[z])$ such that

$$r_s(x,y) = (\phi(x) \otimes \phi(y))r_t(x,y) \qquad \forall x,y \in \mathbb{C}^*,\ x \neq y. \tag{4.19}$$

Therefore, a polynomial equivalence is a regular equivalence that restricts to an automorphism of $\mathfrak{g}[z]$. Quasi-trigonometric r-matrices were introduced and classified up to polynomial equivalence and choice of a maximal order in [26,32]. More precisely, it was shown that quasi-trigonometric solutions are in one-to-one correspondence with certain Lagrangian subalgebras of $\mathfrak{g} \times \mathfrak{g}(\!(z^{-1})\!)$. Embedding the Lagrangian subalgebra, corresponding to a quasi-trigonometric solution r, into some maximal order of $\mathfrak{g} \times \mathfrak{g}(\!(z^{-1})\!)$, the authors of [26,32] obtained a unique quasi-trigonometric solution r_Q (given by a BD quadruple Q) polynomially equivalent to r. In this setting we get the following results:

- The classification theorem for parabolic subalgebras 4.4 together with Theorem 3.3 gives a new proof of the above-mentioned classification of quasi-trigonometric r-matrices;
- In general, a maximal order in which one can embed the Lagrangian subalgebra corresponding to a quasi-trigonometric solution r is not unique. Choosing two different maximal orders we get two different BD quadruples Q and Q' and two polynomially equivalent quasi-trigonometric r-matrices r_Q and $r_{Q'}$. By Theorem 4.4 this equivalence induces an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}$ that fixes the minimal root, i.e. $\vartheta(\widetilde{\alpha}_0) = \widetilde{\alpha}_0$. Therefore, any quasi-trigonometric solution is polynomially equivalent to exactly one quasi-trigonometric r-matrix r_Q, for some BD quadruple Q, if and only if $\mathfrak{g}$ is of type A_1, B_n, C_n, F_4, G_2 or E_8.
  We note that there exist regularly equivalent quasi-trigonometric solutions r_Q and $r_{Q'}$ which are not polynomially equivalent (see Fig. 1). Therefore regular equivalence is strictly weaker than polynomial one;
- It was shown in [26] that for any quasi-trigonometric r-matrix r there exists a holomorphic function $\phi\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

  $$(\phi(x)^{-1} \otimes \phi(y)^{-1})r(x,y) = X(x/y),$$

  where X is a trigonometric solution in the Belavin–Drinfeld classification [4]. Combining Lemma 3.2 and Theorem 4.4 we get a general version of this statement with more control over the holomorphic equivalence. Precisely, the trigonometric r-matrix $r_Q^{\sigma(\mathbf{1};|\nu|)}$, by definition, always depends on the quotient of its parameters; in order to obtain it from a σ-trigonometric r-matrix r_t^σ, where the coset of σ is conjugate to $\nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$, it is enough to apply a regular equivalence composed with the regrading to the principal grading:

  $$r_t^\sigma(x,y) \xmapsto{\text{regular eq.}} r_Q^\sigma(x,y) \xmapsto{\text{regrading}} r_Q^{\sigma(\mathbf{1};|\nu|)}(x,y) = X(x/y);$$

- Conjecture 1 in [8] is justified: Combining (3.3) with (4.1) we get the explicit formula for a quasi-trigonometric solution r_Q given by a BD quadruple Q, namely

  $$\begin{aligned} r_Q(x,y) &= \frac{yC}{x-y} + \frac{C_{\mathfrak{h}}}{2} + C_- + t_{\mathfrak{h}} + \sum_{\widetilde{\alpha}\in\Phi_1^+}\sum_{j=1}^{\infty} b_{-\widetilde{\alpha}} \wedge \theta_\gamma^j\,(b_{\widetilde{\alpha}}) \\ &= -\frac{1}{2}\left(\frac{y+x}{y-x}C + \sum_{\widetilde{\alpha}\in\Phi^+} b_{\widetilde{\alpha}} \wedge b_{-\widetilde{\alpha}} - t_{\mathfrak{h}} + \sum_{\widetilde{\alpha}\in\Phi_1^+}\sum_{j=1}^{\infty} \theta_\gamma^j\,(b_{\widetilde{\alpha}}) \wedge b_{-\widetilde{\alpha}}\right). \end{aligned} \tag{4.20}$$

  This formula (up to a sign) coincides with the one conjectured by Burban, Galinat and Stolin in [8];
- Question 2 in [8] is answered: Let Q be a BD quadruple and $h := |\sigma_{(\mathbf{1};1)}|$. The relation between the quasi-trigonometric solution (4.20) and the trigonometric solution

  $$\begin{aligned} X(u-v) = t_{\mathfrak{h}} + \frac{C_{\mathfrak{h}}}{2} + \frac{1}{e^{u-v}-1}\sum_{k=0}^{h-1} e^{\frac{k(u-v)}{h}} C_k^{\sigma(\mathbf{1};1)} &+ \sum_{\substack{\widetilde{\alpha}\in\Phi_1^+\\ \widetilde{\alpha}=(\alpha,k)}}\sum_{j=1}^{\infty} e^{\frac{k(u-v)}{h}} \theta_\gamma^j\,(b_{\widetilde{\alpha}})\,(1) \otimes b_{-\widetilde{\alpha}}(1) \\ &- \sum_{\substack{\widetilde{\alpha}\in\Phi_1^+\\ \widetilde{\alpha}=(\alpha,k)}}\sum_{j=1}^{\infty} e^{\frac{k(v-u)}{h}} b_{-\widetilde{\alpha}}(1) \otimes \theta_\gamma^j\,(b_{\widetilde{\alpha}})\,(1), \end{aligned} \tag{4.21}$$

  given by the same quadruple Q (see [4]), is described by regrading from id to the Coxeter automorphism $\sigma_{(\mathbf{1};1)}$ using Lemma 3.2. More precisely,

  $$\left(e^{u\,\mathrm{ad}(\mu)} \otimes e^{v\,\mathrm{ad}(\mu)}\right) r_Q\left(e^u, e^v\right) = r_Q^{\sigma(\mathbf{1};1)}(e^{u/h}, e^{v/h}) = X(u-v); \tag{4.22}$$

- The formula (4.1) for the twist t_Q and Theorem 4.4 prove the quasi-trigonometric version of Drinfeld's conjecture (see [33,34]). More precisely, any quasi-trigonmetric solution $r_0(x,y) + t(x,y)$, $t \in \mathfrak{g}[x] \otimes \mathfrak{g}[y]$ is polynomially equivalent to a quasi-trigonometric solution of the form $r_0(x,y) + p(x) + q(y)$, where $p, q \in (\mathfrak{g} \otimes \mathfrak{g})[z]$ and $\deg(p), \deg(q) \leq 1$.

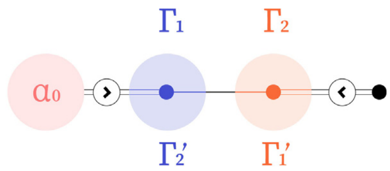


Fig. 1. $\Gamma_1, \Gamma_1', \Gamma_2$ and Γ_2' leading to regularly but not polynomially equivalent r-matrices r_Q and $r_{Q'}$.

4.3. Special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\sigma = \mathrm{id}$

The classification of classical twists of the standard Lie bialgebra structure $\delta_0 := \delta_0^{\mathrm{id}}$ on $\mathfrak{L} = \mathfrak{g}[z, z^{-1}]$ with $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ can be done without heavy geometric machinery. More precisely, using the theory of maximal orders developed in [33], we can show that the equivalence classes of twisted standard bialgebra structures on $\mathfrak{L}$ are in one-to-one correspondence with the equivalence classes of quasi-trigonometric r-matrices, which were classified in [26] in terms of BD quadruples.

The following lemma explains the way in which orders emerge in our work.

Lemma 4.7. *Classical twists t of the standard Lie bialgebra structure δ_0 are in one-to-one correspondence with Lagrangian Lie subalgebras $\widehat{W}_t \subseteq \mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z^{-1})\!)$ satisfying the conditions:*

1. *$\Delta \dotplus \widehat{W}_t = \mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z^{-1})\!)$;*
2. *There are non-negative integers N and M such that*
$$z^N \mathfrak{g}[\![z]\!] \subseteq \pi_1 \widehat{W}_t \subseteq z^{-N} \mathfrak{g}[\![z]\!],$$
$$z^{-M} \mathfrak{g}[\![z^{-1}]\!] \subseteq \pi_2 \widehat{W}_t \subseteq z^{M} \mathfrak{g}[\![z^{-1}]\!],$$
where π_1 and π_2 are the projections of $\mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z^{-1})\!)$ onto its components $\mathfrak{g}(\!(z)\!)$ and $\mathfrak{g}(\!(z^{-1})\!)$ respectively.

Proof. By Remark 3.6 the standard Lie bialgebra structure δ_0 on $\mathfrak{L}$ is defined by the Manin triple
$$(\mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z^{-1})\!), \Delta, \widehat{W}_0). \tag{4.23}$$
Therefore, in view of Theorem 2.4 and its proof, it is enough to show that condition *2.* corresponds to the commensurability condition on $\widehat{W}_t$ and $\widehat{W}_0$, or equivalently, to finite dimensionality of the image of the map $T = \psi(t) \colon \widehat{W}_0 \longrightarrow \Delta$. The latter correspondence is justified by the following chain of arguments: the condition $\dim(\mathrm{im}(T)) < \infty$ is equivalent to the inclusion
$$\mathrm{im}(T) \subseteq \left\{ \left(\sum_{k=-N}^{M} a_k z^k, \sum_{k=-N}^{M} a_k z^k \right) \middle| \, a_k \in \mathfrak{g} \right\}, \tag{4.24}$$
for some non-negative integers N and M, which in its turn is equivalent to
$$\begin{aligned} &\pi_1 \widehat{W}_t \subseteq z^{-N} \mathfrak{g}[\![z]\!], \\ &\pi_2 \widehat{W}_t \subseteq z^{M} \mathfrak{g}[\![z^{-1}]\!]; \end{aligned} \tag{4.25}$$
Since $\widehat{W}_t$ is Lagrangian, inclusions (4.25) are equivalent to condition *2.* of the theorem. ■

A subalgebra $W \subseteq \mathfrak{g}(\!(u)\!)$ is called an *order* if there is a non-negative integer N such that
$$u^N \mathfrak{g}[\![u]\!] \subseteq W \subseteq u^{-N} \mathfrak{g}[\![u]\!]. \tag{4.26}$$
Therefore, condition *2.* of Lemma 4.7 means that the projections $\pi_1 \widehat{W}_t$ and $\pi_2 \widehat{W}_t$ are orders.

The following two results from [33] play the key role in the classification of classical twists of δ_0.

Theorem 4.8. *For any order W in $\mathfrak{sl}(n, \mathbb{C}(\!(u^{-1})\!))$ there is a matrix $A \in \mathrm{GL}(n, \mathbb{C}(\!(u^{-1})\!))$ such that $W \subseteq A^{-1} \mathfrak{sl}(n, \mathbb{C}[\![u^{-1}]\!]) A$. In particular, any maximal order must be of the form $A^{-1} \mathfrak{sl}(n, \mathbb{C}[\![u^{-1}]\!]) A$ for some $A \in \mathrm{GL}(n, \mathbb{C}(\!(u^{-1})\!))$.*

Lemma 4.9 (*Sauvage Lemma*)**.** *The diagonal matrices $\mathrm{diag}(u^{m_1}, \dots, u^{m_n})$, where $m_k \in \mathbb{Z}$ and $m_1 \le \dots \le m_n$, represent all double cosets in $\mathrm{GL}(n, \mathbb{C}[\![u^{-1}]\!]) \backslash \mathrm{GL}(n, \mathbb{C}(\!(u^{-1})\!)) / \mathrm{GL}(n, \mathbb{C}[u])$.*

Let $W \subseteq \mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z^{-1})\!)$ be a Lagrangian subalgebra satisfying the conditions of Lemma 4.7. Since the projections $\pi_1 W \subseteq \mathfrak{g}(\!(z)\!)$ and $\pi_2 W \subseteq \mathfrak{g}(\!(z^{-1})\!)$ are orders, by Theorem 4.8 there are matrices $A_\pm \in GL(n, \mathbb{C}(\!(z^{\pm 1})\!))$ such that
$$\begin{aligned} &\pi_1 W \subseteq A_+^{-1} \mathfrak{sl}(n, \mathbb{C}[\![z]\!]) A_+, \\ &\pi_2 W \subseteq A_-^{-1} \mathfrak{sl}(n, \mathbb{C}[\![z^{-1}]\!]) A_-. \end{aligned} \tag{4.27}$$

By Sauvage Lemma 4.9 we can find matrices

$$\begin{aligned} &P_\pm \in \mathrm{GL}(n, \mathbb{C}[\![z^{\pm 1}]\!]),\\ &d_\pm \in \mathrm{GL}(n, \mathbb{C}[z, z^{-1}]),\\ &Q_\pm \in \mathrm{GL}(n, \mathbb{C}[z^{\mp 1}]), \end{aligned} \tag{4.28}$$

where $d_\pm$ are diagonal, such that

$$\begin{aligned} \pi_1 W &\subseteq Q_+^{-1} d_+^{-1} P_+^{-1} \mathfrak{sl}(n, \mathbb{C}[\![z]\!]) P_+ d_+ Q_+,\\ \pi_2 W &\subseteq Q_-^{-1} d_-^{-1} P_-^{-1} \mathfrak{sl}(n, \mathbb{C}[\![z^{-1}]\!]) P_- d_- Q_-. \end{aligned} \tag{4.29}$$

Taking the product and using the fact that $P_\pm^{-1}\mathfrak{sl}(n, \mathbb{C}[\![z^{\pm 1}]\!])P_\pm = \mathfrak{sl}(n, \mathbb{C}[\![z^{\pm 1}]\!])$ we obtain the inclusion

$$W \subseteq \left(Q_+^{-1} d_+^{-1} \mathfrak{sl}(n, \mathbb{C}[\![z]\!]) d_+ Q_+\right) \times \left(Q_-^{-1} d_-^{-1} \mathfrak{sl}(n, \mathbb{C}[\![z^{-1}]\!]) d_- Q_-\right). \tag{4.30}$$

Note that the componentwise conjugation by Q_+ or d_+ is a regular equivalence. Applying these conjugations we get

$$\begin{aligned} \widetilde{W} := d_+ Q_+ W Q_+^{-1} d_+^{-1} &\subseteq \mathfrak{sl}(n, \mathbb{C}[\![z]\!]) \times \left(d_+ Q_+ Q_-^{-1} d_-^{-1} \mathfrak{sl}(n, \mathbb{C}[\![z^{-1}]\!]) d_- Q_- Q_+^{-1} d_+^{-1}\right)\\ &\subseteq \mathfrak{sl}(n, \mathbb{C}[\![z]\!]) \times \mathfrak{sl}(n, \mathbb{C}(\!(z^{-1})\!)) \end{aligned} \tag{4.31}$$

By Theorem 3.3 the classification problem of classical twists of the standard Lie bialgebra structure δ_0 is equivalent to the classification problem of id-trigonometric r-matrices. The following lemma reduces the question even further to quasi-trigonometric r-matrices.

Lemma 4.10. *Any* id-*trigonometric r-matrix is regularly equivalent to a quasi-trigonometric one.*

Proof. Let $r_t = r_0 + t$ be an id-trigonometric r-matrix, where t is a classical twist of δ_0, and W_t be the corresponding Lagrangian subalgebra of $\mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z^{-1})\!)$. By the argument preceding the lemma there is a regular equivalence $\phi \in \mathrm{Aut}_{\mathbb{C}[z,z^{-1}]-\mathrm{LieAlg}}(\mathfrak{g}[z, z^{-1}])$ such that

$$W_s := (\phi \times \phi) W_t \subseteq \mathfrak{sl}(n, \mathbb{C}[\![z]\!]) \times \mathfrak{sl}(n, \mathbb{C}(\!(z^{-1})\!)). \tag{4.32}$$

for some classical twist s of δ_0. We now show that r_s is quasi-trigonometric, or equivalently, that $s \in \mathfrak{g}[z]^{\otimes 2} \cong \mathfrak{g}^{\otimes 2}[x, y]$. Let $\{I_\alpha\}_{\alpha=1}^n$ be an orthonormal basis for $\mathfrak{sl}(n, \mathbb{C})$. Then we can write

$$s = s_{ij}^{\alpha\beta} I_\alpha x^i \otimes I_\beta y^j. \tag{4.33}$$

Assume that $s_{k\ell}^{\alpha'\beta'} \neq 0$ for some $\alpha', \beta' \in \{1, \dots, n\}$ and $k, \ell \in \mathbb{Z}$ such that at least one of the indices k or ℓ is strictly negative, i.e. the tensor s contains a negative power of z in one of its components. Since s is skew-symmetric we may assume without loss of generality that $k < 0$. Then

$$\pi_1\left(s_{ij}^{\alpha\beta} B\left((I_\beta z^j, I_\beta z^j), (I_{\beta'} z^{-\ell}, 0)\right)(I_\alpha z^i, I_\alpha z^i)\right) = s_{i\ell}^{\alpha\beta'} I_\alpha z^i \tag{4.34}$$

where the sum on the right-hand side contains z^k, $k < 0$. However, by (4.32) the projection $\pi_1(W_s)$ is contained in $\mathfrak{sl}(n, \mathbb{C}[\![z]\!])$ and hence cannot contain negative powers of z. This contradiction shows that both components of s are polynomials in z. ■

Quasi-trigonometric r-matrices over $\mathfrak{sl}(n, \mathbb{C})$ were classified (up to regular equivalence) in [26] using BD quadruples we introduced at the beginning of this section. One can show that if we lift the Lagrangian subalgebra $W \subseteq \mathfrak{g} \times \mathfrak{g}(\!(z^{-1})\!)$, constructed from a BD quadruple Q in [26], to $\mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z^{-1})\!)$ we get precisely the Lagrangian subalgebra $\widehat{W}_Q$ determined by the relation

$$(\mathfrak{L} \times \mathfrak{L}) \cap \widehat{W}_Q = W_Q, \tag{4.35}$$

where W_Q is given by (4.5). By Lemma 4.7 the Lagrangian subalgebra $\widehat{W}_Q$ uniquely determines the classical twist t_Q. This gives the classification of classical twists and proves the first part of Theorem 4.1 in the special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.

Remark 4.11. The statement of Lemma 4.10 is not surprising. Its general version can be deduced from Theorems 4.1 and 4.4. Precisely, for any finite-dimensional simple Lie algebra $\mathfrak{g}$ an id-trigonometric solution $r_Q = r_0 + t_Q$, given by a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$, is regularly equivalent to a quasi-trigonometric one if and only if there is an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}$ such that $\widetilde{\alpha}_0 \notin \vartheta(\Gamma_1)$. It is easy to check that this condition is always satisfied for Dynkin diagrams of types $A_n^{(1)}, C_n^{(1)}, B_{2-4}^{(1)}$ and $D_{4-10}^{(1)}$. Therefore, in these cases any id-trigonometric solution is regularly equivalent to a quasi-trigonometric one. In other cases it is always possible to find a BD quadruple Q, such that r_Q is not equivalent to a quasi-trigonometric r-matrix (see Fig. 2). ◇

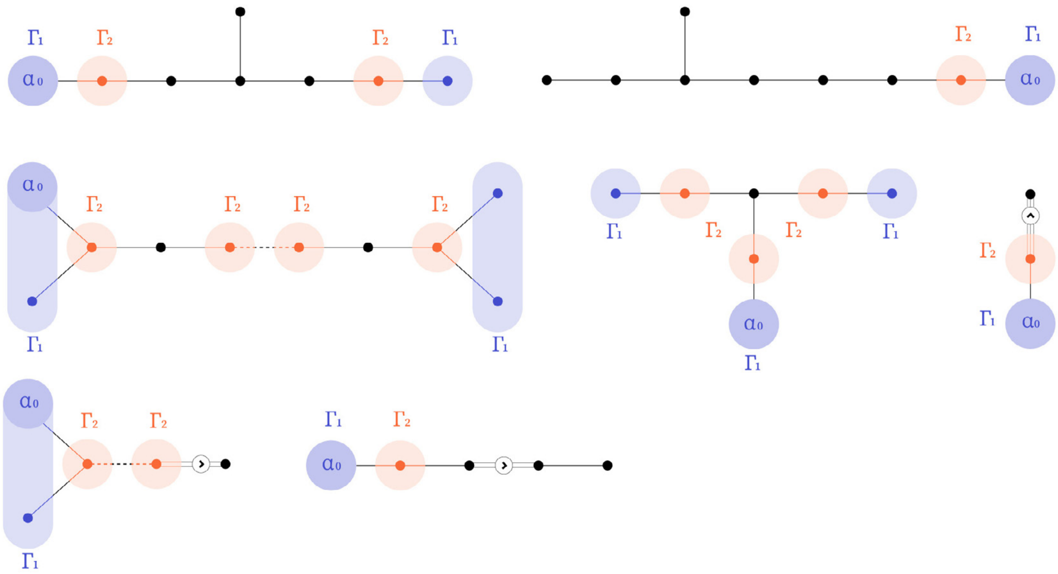


Fig. 2. Examples of Γ_1 and Γ_2 giving rise to id-trigonometric solutions not equivalent to quasi-trigonometric ones. The dashed lines mean any number $m \geq 1$ of vertices.

5. Algebro-geometric proof of the main classification theorem

In this section we give a brief summary of the results in [7], prove the extension property for formal equivalences between geometric r-matrices (see Theorem 5.5) and, finally, combining this property with the results in [1] on geometrization of σ-trigonometric r-matrices we verify Theorem 4.1.

5.1. Survey on the geometric theory of the CYBE

Let E be an irreducible projective curve of arithmetic genus 1. Then E is a *Weierstraß cubic*, i.e. there are parameters $g_2, g_3 \in \mathbb{C}$ such that E is the projective closure of $E_\circ = \mathrm{V}(y^2 - 4x^3 + g_2x + g_3) \subseteq \mathbb{P}^2_{(w:x:y)}$ by a smooth point p at infinity. E is singular if and only if $g_2^3 = 27g_3^2$ and an elliptic curve otherwise. In the singular case it has a unique singular point s, which is a simple cusp if $g_2 = 0 = g_3$ and a simple node otherwise. Let $\breve{E}$ be the set of smooth points of E. Fix a non-zero section $\omega \in \Gamma(E, \Omega_E) \cong \mathbb{C}$, where Ω_E is the dualizing sheaf. We view ω as a global regular 1-form in the Rosenlicht sense (see e.g. [2, Section II.6]).

We consider now a coherent sheaf $\mathcal{A}$ of Lie algebras on E such that

(i) $\mathrm{H}^0(E, \mathcal{A}) = 0 = \mathrm{H}^1(E, \mathcal{A})$ and

(ii) $\breve{\mathcal{A}} = \mathcal{A}|_{\breve{E}}$ is *weakly* $\mathfrak{g}$*-locally free*, i.e. $\mathcal{A}|_p \cong \mathfrak{g}$ as Lie algebras for all $p \in \breve{E}$.

Property (i) gives that $\mathcal{A}$ is torsion free and property (ii) ensures that the rational envelope A_K of the sheaf $\mathcal{A}$ is a simple Lie algebra over the field K of rational functions on E. Together these properties give the existence of a distinguished section, called *geometric r-matrix*, $\rho \in \Gamma(\breve{E} \times \breve{E}, \breve{\mathcal{A}} \boxtimes \breve{\mathcal{A}}(D))$, where $D = \{(x, x) \in \breve{E} \times \breve{E} : x \in \breve{E}\}$ is the *diagonal divisor*. This section satisfies a geometric version of a generalized CYBE, although, if E is singular, it lacks skew-symmetry in general, which prevents it to solve the CYBE. Thus we demand one more property of $\mathcal{A}$ in this case, which ensures skew-symmetry.

If E is singular with a singularity s, we can consider the invariant non-degenerate $\mathbb{C}$-bilinear form

$$B_s^\omega : A_K \times A_K \longrightarrow K \xrightarrow{\mathrm{res}_s^\omega} \mathbb{C}, \tag{5.1}$$

where the first map is the Killing form of A_K over K and $\mathrm{res}_s^\omega(f) = \mathrm{res}_s(f\omega)$ is the residue taken in the Rosenlicht sense.

(iii) $\mathcal{A}_s \subset A_K$ is isotropic, i.e. $B_s^\omega(\mathcal{A}_s, \mathcal{A}_s) = 0$.

Now the main statement of the geometric approach to the CYBE is the following.

Theorem 5.1 ([7, Theorem 4.3]). *The geometric r-matrix ρ is a non-degenerate and skew symmetric (both meant in an appropriate geometric manner) solution of a geometric version of the CYBE.*

We want to describe ρ as a series, which can be thought of as a Taylor expansion in the second coordinate at the smooth point p at infinity. To do so, let us switch from the sheaf theoretic setting to one localized at the formal neighbourhood of p. There is a unique element u inside the $\mathfrak{m}_p$-adic completion $\widehat{O}_p$ of the local ring $(\mathcal{O}_{E,p}, \mathfrak{m}_p)$, such that $u(p) = 0$ and $\widehat{\omega}_p = du$. We can identify $\widehat{O}_p$ with $\mathbb{C}[\![u]\!]$. Thus the field of fractions $\widehat{Q}_p$ can be identified with $\mathbb{C}(\!(u)\!)$. Consequently, we may view $O = \Gamma(E_\circ, \mathcal{O}_E)$ as a subalgebra of $\widehat{Q}_p = \mathbb{C}(\!(u)\!)$.

Since $\mathfrak{g}$ is simple, Whitehead's lemma implies that $\mathrm{H}^2(\mathfrak{g}, \mathfrak{g}) = 0$ and hence all formal deformations of $\mathfrak{g}$ are trivial (see e.g. [19, Section A.8]). Thus $\widehat{\mathcal{A}}_p$, which can be understood as a formal deformation of $\mathfrak{g}$ by Property (ii) of $\mathcal{A}$, is trivial as a formal deformation, i.e. there exists an $\widehat{O}_p = \mathbb{C}[\![u]\!]$-linear isomorphism $\xi\colon \widehat{\mathcal{A}}_p \longrightarrow \mathfrak{g}[\![u]\!]$, called *formal trivialization*, of Lie algebras such that the induced isomorphism

$$\mathfrak{g} \cong \widehat{\mathcal{A}}_p/\mathfrak{m}_p\widehat{\mathcal{A}}_p \longrightarrow \mathcal{A}|_p \cong \mathfrak{g} \tag{5.2}$$

is the identity. We obtain an induced Lie algebra isomorphism $Q(\widehat{\mathcal{A}}_p) = \widehat{\mathcal{A}}_p \otimes_{\widehat{O}_p} \widehat{Q}_p \longrightarrow \mathfrak{g}(\!(u)\!)$ via the $\mathbb{C}(\!(u)\!)$-linear extension of ξ, which we denote by the same symbol. We write the image of $\Gamma(E_\circ, \mathcal{A}) \subseteq Q(\widehat{\mathcal{A}}_p)$ under ξ by $\mathfrak{g}(\rho) \subseteq \mathfrak{g}(\!(u)\!)$.

Note that $\mathfrak{g}(\!(u)\!)$ is equipped with the invariant non-degenerate $\mathbb{C}$-bilinear form

$$B_p(f, g) := \mathrm{res}_0\, [\kappa(f, g)du] \qquad f, g \in \mathfrak{g}(\!(u)\!). \tag{5.3}$$

Theorem 5.2 ([7, *Proposition 6.1 & Theorem 6.4*]). *$(\mathfrak{g}(\!(u)\!), \mathfrak{g}(\rho), \mathfrak{g}[\![u]\!])$ is a Manin triple and the Taylor expansion in the second coordinate with respect to u in p gives an injection*

$$\Gamma(\breve{E} \times \breve{E}, \breve{A} \boxtimes \breve{A}(D)) \longrightarrow (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \tag{5.4}$$

which maps ρ to $\sum_{k=0}^{\infty}\sum_{\ell=1}^{n} f_{k\ell} \otimes y^k b_\ell$, where $\{b_\ell\}$ is a basis of $\mathfrak{g}$ and $\{f_{k\ell}\}$ is the basis of $\mathfrak{g}(\rho) \subseteq \mathfrak{g}(\!(u)\!)$, uniquely determined by $B_p(f_{k\ell}, u^{k'} b_{\ell'}) = \delta_{k\ell}\delta_{k'\ell'}$.

Remark 5.3. Let us clarify what we mean by Taylor expansion in the second coordinate. Let $P_k = \mathrm{Spec}(\widehat{O}_p/\mathfrak{m}_p^k\widehat{O}_p)$ and $\iota_k\colon P_k \longrightarrow E$ be the injection, mapping the closed point of P_k to p. Then we can consider the pull-back with respect to $\mathrm{id}_{\breve{E}\setminus\{p\}} \times \iota_k$ to obtain the morphism

$$\Gamma(\breve{E} \times \breve{E}, \breve{A} \boxtimes \breve{A}(D)) \longrightarrow \Gamma(\breve{E} \setminus \{p\} \times P_k, \breve{A} \boxtimes \breve{A}(D)) \cong \Gamma(\breve{E} \setminus \{p\}, \breve{A}) \otimes \widehat{\mathcal{A}}_p/\mathfrak{m}_p^k\widehat{\mathcal{A}}_p, \tag{5.5}$$

where we have used that $(\breve{E} \setminus \{p\} \times P_k \cap D) = \emptyset$ and applied the Künneth isomorphism. Mapping $\Gamma(\breve{E} \setminus \{p\}, \breve{A})$ via ξ to $\mathfrak{g}(\!(u)\!)$, using $\widehat{\mathcal{A}}_p/\mathfrak{m}_p^k\widehat{\mathcal{A}}_p \cong \mathfrak{g}[u]/u^k\mathfrak{g}[u]$ and applying the projective limit with respect to k, yields the desired injection $\Gamma(\breve{E} \times \breve{E}, \breve{A} \boxtimes \breve{A}(D)) \longrightarrow (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$. $\diamond$

The theorem suggests, that the geometric r-matrix ρ actually determines $\mathcal{A}$ completely. Our next goal is to formalize this idea. The construction we present is known in other situations, see e.g. [30]. The algebras O and $\mathfrak{g}(\rho)$ inherit the ascending filtrations from the natural filtrations of $\mathbb{C}(\!(u)\!)$ and $\mathfrak{g}(\!(u)\!)$, namely we have $O_j := O \cap u^{-j}\mathbb{C}[\![u]\!]$, $\mathfrak{g}(\rho)_j := \mathfrak{g}(\rho) \cap u^{-j}\mathfrak{g}[\![u]\!]$ and

$$\ldots = 0 = O_{-1} \subseteq \mathbb{C} = O_0 \subseteq O_1 \subseteq \ldots, \qquad \ldots = 0 = \mathfrak{g}(\rho)_0 \subseteq \mathfrak{g}(\rho)_1 \subseteq \mathfrak{g}(\rho)_2 \subseteq \ldots, \tag{5.6}$$

such that $O_jO_k \subseteq O_{k+j}$, $O_j\mathfrak{g}(\rho)_k \subseteq \mathfrak{g}(\rho)_{j+k}$ and $[\mathfrak{g}(\rho)_j, \mathfrak{g}(\rho)_k] \subseteq \mathfrak{g}(\rho)_{j+k}$. Therefore, we can consider the associated graded objects[6] $\mathrm{gr}(O)$ and $\mathrm{gr}(\mathfrak{g}(\rho))$, given by

$$\mathrm{gr}(O) := \bigoplus_{j=0}^{\infty} O_j \;\text{ and }\; \mathrm{gr}(\mathfrak{g}(\rho)) := \bigoplus_{j=0}^{\infty} \mathfrak{g}(\rho)_j. \tag{5.7}$$

Note that $\mathrm{gr}(\mathfrak{g}(\rho))$ is a graded Lie algebra over the graded $\mathbb{C}$-algebra $\mathrm{gr}(O)$. Let us denote by $\mathrm{gr}(\mathfrak{g}(\rho))^\sim$ the associated quasi-coherent sheaf of Lie algebras on $\mathrm{Proj}(\mathrm{gr}(O))$ (see e.g. [18, §II.5]).

Lemma 5.4. *We have $E = \mathrm{Proj}(\mathrm{gr}(O))$ and the formal trivialization ξ induces an isomorphism $\mathcal{A} \longrightarrow \mathrm{gr}(\mathfrak{g}(\rho))^\sim$ of sheaves of Lie algebras, which we again denote by ξ.*

Proof. We can view $E_\circ = \mathrm{Spec}(O)$ as an affine open subscheme of $\mathrm{Proj}(\mathrm{gr}(O))$ by identifying any prime ideal $\mathfrak{p}$ of O (which inherits a natural filtration) with $\mathrm{gr}(\mathfrak{p})$. Under this identification we have $\Gamma(E_\circ, \mathrm{gr}(\mathfrak{g}(\rho))^\sim) = \mathfrak{g}(\rho)$, which is most easily seen by using the definitions in [18, §II.5]. In particular, $\xi\colon \Gamma(E_\circ, \mathcal{A}) \longrightarrow \mathfrak{g}(\rho) = \Gamma(E_\circ, \mathrm{gr}(\mathfrak{g}(\rho))^\sim)$ is an isomorphism of Lie algebras over O.

By [31, Proposition 3] for any coherent sheaf $\mathcal{F}$ on an open neighbourhood U of p the sequence

$$0 \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U \setminus \{p\}, \mathcal{F}) \oplus \hat{\mathcal{F}}_p \longrightarrow \hat{\mathcal{F}}_p \otimes_{\widehat{O}_p} \widehat{Q}_p \tag{5.8}$$

[6] These should not be confused with the graded objects associated to modules with a descending filtration.

is exact. This implies $\Gamma(U,\mathcal{F}) = \Gamma(U\setminus\{p\},\mathcal{F})\cap\hat{\mathcal{F}}_p$.

Thus, for any $a\in O_j\setminus O_{j-1}$ we have that $\mathrm{D}(a)\cup\{p\}$ is an affine (see [16, Proposition 5]) open neighbourhood of p in E with the coordinate ring $O[a^{-1}]\cap\mathbb{C}[\![u]\!] = (\mathrm{gr}(O)[a^{-1}])_0$, where $a\in\mathrm{gr}(O)$ is taken to have degree j. Thus we have a natural identification of affine schemes $E\supseteq \mathrm{D}(a)\cup\{p\} = \mathrm{D}_+(a)\subseteq \mathrm{Proj}(\mathrm{gr}(O))$ and we see that $E\subseteq\mathrm{Proj}(\mathrm{gr}(O))$. Since now E is a projective curve identified with an open subset in the projective curve $\mathrm{Proj}(\mathrm{gr}(O))$, [16, Proposition 1] implies the equality $E=\mathrm{Proj}(\mathrm{gr}(O))$.

Finally, by definition we have $\Gamma(\mathrm{D}_+(a),\mathrm{gr}(\mathfrak{g}(\rho))^{\sim}) = \Gamma(\mathrm{D}(a)\cup\{p\},\mathrm{gr}(\mathfrak{g}(\rho))^{\sim}) = (\mathfrak{g}(\rho)[a^{-1}])_0 = \mathfrak{g}(\rho)[a^{-1}]\cap\mathfrak{g}[\![u]\!]$ and hence we obtain the isomorphism

$$\xi\colon \Gamma(\mathrm{D}(a)\cup\{p\},\mathcal{A}) = \Gamma(\mathrm{D}(a),\mathcal{A})\cap\hat{\mathcal{A}}_p \longrightarrow \mathfrak{g}(\rho)[a^{-1}]\cap\mathfrak{g}[\![u]\!] = \Gamma(\mathrm{D}(a)\cup\{p\},\mathrm{gr}(\mathfrak{g}(\rho))^{\sim}).$$

This ends the proof, because $E=\mathrm{Proj}(\mathrm{gr}(O)) = E_\circ\cup\mathrm{D}_+(a)$. ∎

5.2. Extension property of formal local equivalences

Now let us consider two coherent sheaves of Lie algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ on E satisfying the conditions (i) - (iii) of Section 5.1 and denote by ρ_1 and ρ_2 the corresponding geometric r-matrices. Fix formal trivializations ξ_i of $\mathcal{A}_i$ at p and consider the corresponding isomorphisms $\xi_i\colon\mathcal{A}_i \longrightarrow \mathrm{gr}(\mathfrak{g}(\rho_i))^{\sim}$ for $i=1,2$, where $\mathfrak{g}(\rho_i) = \mathrm{Span}_{\mathbb{C}}(\{f_{k\ell}^{(i)}\})\subseteq\mathfrak{g}(\!(u)\!)$ is the image of $\Gamma(E_\circ,\mathcal{A}_i)$ and

$$\sum_{k=0}^{\infty}\sum_{\ell=1}^{n} f_{k\ell}^{(i)}\otimes y^k b_\ell\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] \tag{5.9}$$

are the Taylor expansions of ρ_i described in Theorem 5.2. We are now in a position to show that any formal equivalence of ρ_1 and ρ_2 at p extends to a global isomorphism of the corresponding sheaves.

Theorem 5.5. *Let $\phi\colon\mathfrak{g}[\![u]\!]\longrightarrow\mathfrak{g}[\![u]\!]$ be a $\mathbb{C}[\![u]\!]$-linear automorphism of Lie algebras such that*

$$\sum_{k=0}^{\infty}\sum_{\ell=1}^{n}\phi(f_{k\ell}^{(1)})\otimes\phi(y^k b_\ell) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{n} f_{k\ell}^{(2)}\otimes y^k b_\ell, \tag{5.10}$$

where we consider the $\mathbb{C}(\!(u)\!)$-linear expansion of ϕ in the first tensor factor. Then there is an isomorphism $\psi\colon\mathcal{A}_1\longrightarrow\mathcal{A}_2$ of coherent sheaves of Lie algebras such that $\xi_2\hat{\psi}_p\xi_1^{-1}=\phi$ and $(\psi\boxtimes\psi)\rho_1=\rho_2$, where we consider the linear extension with respect to the rational functions on $\breve{E}\times\breve{E}$.

Proof. Write $\phi=\sum_{j=0}^{\infty}u^j\phi_j\in\mathrm{End}(\mathfrak{g})[\![u]\!]$. Then

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{\ell=1}^{n}\phi(f_{k\ell}^{(1)})\otimes\phi(y^k b_\ell) &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{n}\sum_{j=0}^{k}\phi(f_{(k-j)\ell}^{(1)})\otimes y^k\phi_j(b_\ell)\\ &= \sum_{k=0}^{\infty}\sum_{\ell'=1}^{n}\left(\sum_{\ell=1}^{n}\sum_{j=0}^{k}a_{\ell'\ell}^{j}\phi(f_{(k-j)\ell}^{(1)})\right)\otimes y^k b_{\ell'},\end{aligned} \tag{5.11}$$

where $\phi_j(b_\ell)=\sum_{\ell'=1}^{n}a_{\ell'\ell}^{j}b_{\ell'}$. This shows that $\mathfrak{g}(\rho_2)\subseteq\phi(\mathfrak{g}(\rho_1))$ by comparing coefficients in (5.10). Since $\mathfrak{g}[\![u]\!]\dotplus\mathfrak{g}(\rho_2) = \mathfrak{g}[\![u]\!]\dotplus\phi(\mathfrak{g}(\rho_1))$, we see that $\mathfrak{g}(\rho_2)=\phi(\mathfrak{g}(\rho_1))$.

Clearly the $\mathbb{C}(\!(u)\!)$-linear extension of ϕ preserves the filtration of $\mathfrak{g}(\!(u)\!)$ and hence induces a graded isomorphism of Lie algebras $\phi\colon\mathrm{gr}(\mathfrak{g}(\rho_1))\longrightarrow\mathrm{gr}(\mathfrak{g}(\rho_2))$. Since the procedure $(\cdot)^{\sim}$ of associating a quasi-coherent sheaf to a graded module on $E=\mathrm{Proj}(\mathrm{gr}(O))$ is functorial, we get an isomorphism $\phi^{\sim}\colon\mathrm{gr}(\mathfrak{g}(\rho_1))^{\sim}\longrightarrow\mathrm{gr}(\mathfrak{g}(\rho_2))^{\sim}$. Thus we can define $\psi:=\xi_2^{-1}\phi^{\sim}\xi_1\colon\mathcal{A}_1\longrightarrow\mathcal{A}_2$. Applying [15, Lemma 1.10] we see that $(\psi\boxtimes\psi)\rho_1=\rho_2$. ∎

Remark 5.6. Since the induced isomorphisms in (5.2) of the formal trivializations ξ_i ($i=1,2$) give the identity, we have $\phi_0=\psi|_p$. ◇

5.3. Proof of the main classification theorem

Fix an automorphism $\sigma\in\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ of finite order m and an outer automorphism ν from the coset $\sigma\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. We want to apply Theorem 5.5 to reduce the statement of Theorem 4.1 to holomorphic equivalences. Therefore, we need sheaves which give rise to σ-trigonometric r-matrices. These were constructed in [1].

Theorem 5.7 ([1, Theorem 6.9]). *Let t be a classical twist of δ_0^{σ}, E be a nodal Weierstraß cubic with global nonvanishing 1-form $\omega=d(z^m)/z^m$ under the identification $\breve{E}=\mathrm{Spec}(\mathbb{C}[z^m,z^{-m}])$. Then there exists a coherent sheaf of Lie algebras $\mathcal{A}_t$ on E, satisfying properties (i)-(iii) of Section 5.1, such that*

1. $\Gamma(\breve{E}, \mathcal{A}_t) = \mathfrak{L}^\sigma$ and
2. the isomorphism

$$\Gamma(\breve{E} \otimes \breve{E}, \breve{\mathcal{A}}_t \boxtimes \breve{\mathcal{A}}_t(D)) \cong \left(\frac{1}{(x/y)^m - 1}\right) \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \tag{5.12}$$

maps the geometric r-matrix ρ_t of $\mathcal{A}_t$ to r_t^σ.

We can use this statement to apply the extension of equivalence scheme presented in the last subsection.

Theorem 5.8. *Let r_t^σ and r_s^σ be σ-trigonometric r-matrices. There exists an open neighbourhood U of 0 and a holomorphic function $\varphi: U \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ satisfying*

$$(\varphi(u) \otimes \varphi(v))r_t(e^{u/m}, e^{v/m}) = r_s(e^{u/m}, e^{v/m}) \tag{5.13}$$

if and only if r_t^σ and r_s^σ are regularly equivalent.

Proof. If r_t^σ and r_s^σ are regularly equivalent via ϕ, we have $(\phi(x) \otimes \phi(y))r_t^\sigma(x, y) = r_s^\sigma(x, y)$, therefore equation Eq. (5.13) is satisfied for $\varphi(z) := \phi(e^{z/m})$.

It remains to show the other direction. Let $\mathcal{A}_t$ and $\mathcal{A}_s$ be the sheaves on a nodal Weierstraß cubic E provided by Theorem 5.7 for the classical twists t and s of δ_0^σ. We may identify the smooth point p at infinity with $(z^m - 1) \in \breve{E} = \mathrm{Spec}(\mathbb{C}[z^m, z^{-m}])$. The algebra homomorphism $\mathbb{C}[z^m, z^{-m}] \longrightarrow \mathbb{C}[\![u]\!]$ given by

$$z^m \longmapsto e^u = \sum_{k=0}^{\infty} \frac{u^n}{n!}$$

induces an identification $\widehat{\mathcal{O}}_{E,p} = \mathbb{C}[\![u]\!]$ in such a way that $u(p) = 0$ and $\widehat{\omega}_p = e^{-u}d(e^u) = du$. Thus u is the formal local parameter used in the setting of Theorem 5.2 and onwards. Now the $\mathbb{C}[z^m, z^{-m}]$-$\mathbb{C}[\![u]\!]$-equivariant Lie algebra morphism $\mathfrak{L}^\sigma \longrightarrow \mathfrak{g}[\![u]\!]$, given by $f(z) \longmapsto f(e^{u/m})$, induces formal trivializations $\widehat{\mathcal{A}}_p \longrightarrow \mathfrak{g}[\![u]\!]$ for $\mathcal{A} \in \{\mathcal{A}_t, \mathcal{A}_s\}$. With this choice, we can interpret the series expansion of ρ_t and ρ_s, described in Remark 5.3, as the Taylor series of $r_t^\sigma(e^{u/m}, e^{v/m})$ and $r_s^\sigma(e^{u/m}, e^{v/m})$ at $v = 0$. Therefore, we can apply Theorem 5.5 to the Taylor expansion of Eq. (5.15) at $v = 0$ by understanding the Taylor series of φ in 0 as a $\mathbb{C}[\![u]\!]$-linear automorphism of $\mathfrak{g}[\![u]\!]$ to obtain an isomorphism $\psi: \mathcal{A}_t \longrightarrow \mathcal{A}_s$, satisfying $(\psi \boxtimes \psi)\rho_t = \rho_s$.

We obtain a regular equivalence $\psi: \mathfrak{L}^\sigma = \Gamma(\breve{E}, \mathcal{A}_t) \longrightarrow \Gamma(\breve{E}, \mathcal{A}_s) = \mathfrak{L}^\sigma$ by applying $\Gamma(\breve{E}, -)$. Using the commutative diagram

$$\begin{tikzcd} \Gamma(\breve{E} \times \breve{E}, \breve{\mathcal{A}}_t \boxtimes \breve{\mathcal{A}}_t) \arrow[r, "\psi\boxtimes\psi"] \arrow[d, "\cong"'] & \Gamma(\breve{E} \times \breve{E}, \breve{\mathcal{A}}_s \boxtimes \breve{\mathcal{A}}_s) \arrow[d, "\cong"] \\ \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \arrow[r, "\psi\otimes\psi"'] & \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \end{tikzcd}$$

where the vertical isomorphisms are given by the Künneth theorem, we obtain the desired identity

$$(\psi(x) \otimes \psi(y))r_t^\sigma(x, y) = r_s^\sigma(x, y). \quad \blacksquare \tag{5.14}$$

Combining Theorem 5.8 with the results of Section 3.1 and [4], we get the following statement.

Corollary 5.9. *Let $t \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma$ be a classical twist of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. There exists a BD quadruple Q such that r_t^σ is regularly equivalent to r_Q^σ*

Proof. By Theorem 5.8 it is sufficient to show that there exists a holomorphic function $\phi: \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\phi(u) \otimes \phi(v))r_t^\sigma(e^{u/m}, e^{v/m}) = r_Q^\sigma(e^{u/m}, e^{v/m}). \tag{5.15}$$

By Theorem 3.4 and its proof there exists a holomorphic function $\phi_1: \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a trigonometric (in the sense of the Belavin–Drinfeld classification) r-matrix X such that

$$\phi_1(0) = \mathrm{id}_\mathfrak{g} \ \text{ and } \ X(u - v) = (\phi_1(u) \otimes \phi_1(v))r_t^\sigma(e^{u/m}, e^{v/m}). \tag{5.16}$$

Furthermore, it is shown in [4, Theorem 6.1] that there is a holomorphic function $\phi_2: \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$\phi_2(0) = \mathrm{id}_\mathfrak{g} \ \text{ and } \ (\phi_2(u) \otimes \phi_2(v))X(u - v) = r_Q^{\sigma(\mathbf{1};\mathrm{ord}(\nu))}(e^{u/h}, e^{v/h}), \tag{5.17}$$

where $h := |\sigma_{(\mathbf{1},\mathrm{ord}(\nu))}|$. Combining these results and applying the regrading scheme from Lemma Lemma 3.2 we get the desired holomorphic function. $\blacksquare$

We finish the proof of the main theorem by explaining when two BD quadruples give rise to equivalent twisted standard structures. By virtue of Theorem 5.8 we can formulate the result using holomorphic equivalences.

Theorem 5.10. *Let $Q=(\Gamma_1,\Gamma_2,\gamma,t_{\mathfrak{h}})$ and $Q'=(\Gamma_1',\Gamma_2',\gamma',t_{\mathfrak{h}}')$ be two BD quadruples. Then there exists an open neighbourhood U of 0 and a holomorphic function $\varphi\colon U\to \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ satisfying $r_Q(e^{u/m},e^{v/m})=(\varphi(u)\otimes\varphi(v))r_{Q'}(e^{u/m},e^{v/m})$ if and only if there exists an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(Q)=Q'$.*

Proof. To simplify the notations we assume $\sigma=\sigma_{(s;|\nu|)}$ for $s=(s_0,\dots,s_n)$. The general result follows from Remark 2.7. By Theorem 5.8, there exists an open neighbourhood U of 0 and a holomorphic function $\varphi\colon U\to\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ satisfying $r_Q(e^{u/m},e^{v/m})=(\varphi(u)\otimes\varphi(v))r_{Q'}(e^{u/m},e^{v/m})$ if and only if r_Q and $r_{Q'}$ are regularly equivalent, say by a regular equivalence ϕ.

" $\Longrightarrow$ ": The equation (3.23) implies $\phi R_Q\phi^{-1}=R_{Q'}$. If GE_0 and GE_0' are the generalized eigenspaces of R_Q and $R_{Q'}$ respectively, corresponding to the common eigenvalue 0, then $\phi(GE_0)=GE_0'$. Since both θ_γ^+ and $\theta_{\gamma'}^+$ are nilpotent, the identity (4.2) implies the equality of normalizers $\mathrm{N}_{\mathfrak{L}^\sigma}(GE_0)=\mathrm{N}_{\mathfrak{L}^\sigma}(GE_0')=\mathfrak{B}_+$. Therefore, ϕ is an automorphism of $\mathfrak{L}^\sigma$ fixing the Borel subalgebra $\mathfrak{B}_+$. From the third part of Lemma 4.6, we see that ϕ induces an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$. The identity $\vartheta(Q)=Q'$ follows from Theorem 3.7 and formula (4.5).

" $\Longleftarrow$ ": Let ϑ be a Dynkin diagram automorphism, such that $\vartheta(Q)=Q'$. We want to show that ϑ defines a regular equivalence ϕ on $\mathfrak{L}^\sigma$, such that $\phi(\mathfrak{L}^\sigma_{(\alpha,k)})=\mathfrak{L}^\sigma_{\vartheta(\alpha,k)}$ for any root (α,k) of $\mathfrak{L}^\sigma$. Let ϕ' be the automorphism defined by $\phi'(z^{\pm s_i}X_i^\pm(1)):=z^{\pm s_{\vartheta(i)}}X^\pm_{\vartheta(i)}(1)$. Then it maps the root spaces onto each other in the desired way. We now adjust ϕ' to be $\mathbb{C}[z^m,z^{-m}]$-linear. By [23, Lemma 8.6] we have the equality $\phi'((z^m-1)\mathfrak{L}^\sigma)=((z/a)^m-1)\mathfrak{L}^\sigma$ for some $a\in\mathbb{C}^*$. Let μ_a be the automorphism of $\mathfrak{L}^\sigma$ given by $\mu_a(f(z)):=f(az)$. Note that it preserves the root spaces of $\mathfrak{L}^\sigma$. Define $\phi:=\mu_a\phi'$, then $\phi((z^m-1)\mathfrak{L}^\sigma)=(z^m-1)\mathfrak{L}^\sigma$ and thus

$$\phi(z^{\pm s_i+m}X_i^\pm(1))=z^m\phi(z^{\pm s_i}X_i^\pm(1)) \tag{5.18}$$

implying the $\mathbb{C}[z^m,z^{-m}]$-linearity. The defining relation (3.1) pushed forward to $\mathfrak{L}^\sigma$ now implies that $(\phi\otimes\phi)\delta_Q^\sigma=\delta_{Q'}^\sigma\phi$, hence by Theorem 3.7 we see that r_Q and $r_{Q'}$ are regularly equivalent. ■

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Appendix. Application of geometry to rational r-matrices

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. The Lie algebra $\mathfrak{g}[z]$ has a natural Lie bialgebra structure $\delta\colon\mathfrak{g}[z]\longrightarrow\mathfrak{g}[x]\otimes\mathfrak{g}[y]$ defined by the formula

$$\delta(p)(x,y)=\left[p(x)\otimes 1+1\otimes p(y),\frac{C}{x-y}\right], \tag{A.1}$$

where $p\in\mathfrak{g}[z]$ and $C\in\mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir element. Classical twists of this standard Lie bialgebra structure are in bijection with *rational r-matrices*, i.e. r-matrices of the following form

$$r_t(x,y):=\frac{C}{x-y}+t(x,y), \tag{A.2}$$

where $t\in\mathfrak{g}[x]\otimes\mathfrak{g}[y]$. The corresponding Manin triple structure is described in the following theorem.

Theorem A.1 (*[34, Theorem 1]*)**.** *There is a one-to-one correspondence between rational solutions r_t and Lagrangian Lie subalgebras $W\subseteq\mathfrak{g}(\!(z^{-1})\!)$ such that*

1. $\mathfrak{g}[z]\dotplus W=\mathfrak{g}(\!(z^{-1})\!)$ *and*
2. $z^{-N}\mathfrak{g}[z^{-1}]\subseteq W$ *for some* $N>0$.

Moreover, two rational solutions r_t and r_s are (polynomialy) equivalent, i.e.

$$r_s(x,y)=(\phi(x)\otimes\phi(y))r_t(x,y)$$

for some $\phi\in\mathrm{Aut}_{\mathbb{C}[z]-\mathrm{LieAlg}}(\mathfrak{g}[z])$, if and only if $W_s=\phi W_t$. □

In other words, rational solutions are in bijection with Manin triples of the form $(\mathfrak{g}(\!(z^{-1})\!),\mathfrak{g}[z],W)$, where W is commensurable with $z^{-1}\mathfrak{g}[\![z]\!]$. This theorem can also be considered as a special case of Theorem 2.4. The following result was proven in [7].

Theorem A.2 (*[1, Theorem 6.9]*)**.** *Let r_t be a rational r-matrix, E be a cuspidal Weierstraß cubic with global nonvanishing 1-form $\omega=dz$ under the identification $\breve{E}=\mathrm{Spec}(\mathbb{C}[z])$. Then there exists a coherent sheaf of Lie algebras $\mathcal{A}_t$ on E, satisfying properties (i)-(iii) of Section 5.1, such that*

1. *$\Gamma(\breve{E},\mathcal{A}_t)=\mathfrak{g}[z]$ and*
2. *the isomorphism*

$$\Gamma(\breve{E}\otimes\breve{E},\breve{\mathcal{A}}_t\boxtimes\breve{\mathcal{A}}_t(D))\cong\left(\frac{1}{x-y}\right)\mathfrak{g}[x]\otimes\mathfrak{g}[y] \tag{A.3}$$

maps the geometric r-matrix ρ of $\mathcal{A}_t$ to r_t. □

We may identify the smooth point p at infinity with $(z)\in\breve{E}=\mathrm{Spec}(\mathbb{C}[z])$. Then the inclusion $\mathbb{C}[z]\subset\mathbb{C}[\![z]\!]$ induces an identification $\widehat{\mathcal{O}}_{E,p}=\mathbb{C}[\![z]\!]$ such that the formal local parameter u used in the setting of Theorem 5.5 can by identified with z. Similarly, for every rational r-matrix r_t the inclusion $\Gamma(\breve{E},\mathcal{A}_t)=\mathfrak{g}[z]\subset\mathfrak{g}[\![z]\!]$ induces a formal trivialization $\widehat{\mathcal{A}}_{t,p}\longrightarrow\mathfrak{g}[\![z]\!]$, where $\mathcal{A}_t$ is the sheaf described in Theorem A.2. Therefore, the same line of arguments as in the proof of Theorem 5.8 yields the following result.

Theorem A.3. *Let r_t and r_s be locally holomorphically equivalent rational r-matrices, i.e. there exists an open neighbourhood U of 0 and a holomorphic function $\varphi\colon U\longrightarrow\mathrm{Aut}_{\mathbb{C}-\mathrm{LiaAlg}}(\mathfrak{g})$ such that*

$$r_s(x,y)=(\varphi(x)\otimes\varphi(y))r_t(x,y).$$

Then there exists $\psi\in\mathrm{Aut}_{\mathbb{C}[z]-\mathrm{LieAlg}}(\mathfrak{g}[z])$ such that

$$\psi|_U=\varphi \ \text{ and } \ r_s(x,y)=(\psi(x)\otimes\psi(y))r_t(x,y),$$

where ψ is considered as regular function $\mathbb{C}\longrightarrow\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$.

This statement shows that two different notions of gauge equivalence for rational solutions in the works [33,34] by Stolin and [4] by Belavin and Drinfeld actually coincide. More formally, we have the following corollary.

Corollary A.4. *For any rational r-matrix r_t there exists a function $X\colon\mathbb{C}\longrightarrow\mathfrak{g}\otimes\mathfrak{g}$ of the form*

$$X(z)=\frac{C}{z}+p(z),\ \ p\in(\mathfrak{g}\otimes\mathfrak{g})[z],$$

and an automorphism $\psi\in\mathrm{Aut}_{\mathbb{C}[z]-\mathrm{LieAlg}}(\mathfrak{g}[z])$ such that

$$X(x-y)=(\psi(x)\otimes\psi(y))r_t(x,y),$$

where ψ is considered as regular function $\mathbb{C}\longrightarrow\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$.

Proof. Let $r_t(x,y)=C/(x-y)+t(x,y)$. Following the argument in [3] we can find an open neighbourhood $U\subseteq\mathbb{C}$ of 0 and a holomorphic function $\varphi_1\colon U\longrightarrow\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\varphi_1(x)\otimes\varphi_1(y))r_t(x,y)=:X(x-y).$$

Analyzing the construction of φ_1 as in [26, Proof of Theorem 11.3] and using the fact that $t(x,x)$ is defined on the whole $\mathbb{C}$ we see that φ_1 is defined on the whole $\mathbb{C}$ as well. This implies that $X\colon\mathbb{C}\longrightarrow\mathfrak{g}\otimes\mathfrak{g}$ has a unique pole at 0 with residue C. Therefore, the Belavin–Drinfeld trichotomy result [4, Theorem 1.1] guarantees the existence of a holomorphic function $\varphi_2\colon\mathbb{C}\longrightarrow\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\varphi_2(x)\otimes\varphi_2(y))X(x-y)=\frac{C}{x-y}+p(x-y)=:r_p(x,y),$$

where $p\in\mathfrak{g}[x]\otimes\mathfrak{g}[y]$. Applying Theorem A.3 to r_t, r_p and $\varphi:=\varphi_2\varphi_1$ we get the desired polynomial equivalence ψ. ■

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